ON A PROPERTY OF SEMISIMPLE ALGEBRAIC GROUPS

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Let k be an arbitrary field, G a semisimple algebraic group defined over k. Moreover, let P be a parabolic subgroup in G for which the factor space G/P is defined over k. Let $\mathrm{ind}_k G$ ($\mathrm{ind}_k G/P$, respectively) denote the greatest common divisor of the number [K : k], where K runs through those separable extensions of the field k over which G is decomposable [$(G/P)_K \neq \emptyset$, respectively].

If P is the maximal parabolic subgroup (and G/P is defined over k), then the group of divisor classes of the manifold G/P is isomorphic to Z, and the linear system corresponding to the unit class has the natural structure of the Severy-Brauer manifold over k; we let A(P) denote the corresponding central algebra with division. Let $\mathrm{ind}_k A(P)$ denote the index (i.e., the square root of the dimensionality) of the algebra A(P). Evidently there holds

PROPOSITION 1. $\mathrm{ind}_k A(P) \mid \mathrm{ind}_k G/P$.

If G is an interior form of type A_n, then G is a group of elements with norm 1 in some associative algebra A. If m is the index of the algebra A, and P_i is the maximal parabolic subgroup corresponding to the i-th simple root (in the natural order associated with the Dynkin diagram), then it follows from the theory of associative algebras that $m/(i, n+1) \mid \mathrm{ind}_k G/P_i$. If G is an orthogonal group over k, char k ≠ 2, and P is a parabolic subgroup corresponding to a natural linear representation, then according to the Springer theorem [4], $\mathrm{ind}_k G/P = 2$.

These results lead to the following hypotheses:

Hypothesis 1. $\mathrm{ind}_k G = 1 \Leftrightarrow$ G is decomposable over k.

Hypothesis 2. $\mathrm{ind}_k G/P = 1 \Leftrightarrow (G/P)_k \neq \emptyset$.

Hypothesis 3. If $\mathrm{ind}_k G = m$, then G has a decomposition field of degree m.

Let us note that the reformulation of hypotheses 1 and 2 in the terminology of Galois cohomology is an extension of the known property of group cohomology to the nonabelian case: the homomorphism of a constraint on a Sylow p-group is injective on the p-component of a cohomology group (see [1], n° XII, 10.1). Let us also note that for arbitrary classical groups, hypotheses 1, 2 follow from the Weil results [6] and the above-mentioned properties of orthogonal groups and algebras with division.

LEMMA. If K is a separable extension of the field k, then $\mathrm{ind}_K G \mid \mathrm{ind}_k G$ and $\mathrm{ind}_K G/P \mid \mathrm{ind}_k G/P$.

The proof of this lemma utilizes only the following property of the set W of all separable decomposition fields of the group G (analogously for G/P). If $K \in W$ and L is separable over K, then $L \in W$, if $K \in W$ and K^σ is the conjugate field to K, then $K^\sigma \in W$.

Using the lemma, Proposition 1, the results presented above on algebras with division and on orthogonal groups, as well as some special properties of the groups under consideration, we can prove the following theorem.

THEOREM. Let char k > rg G. The assertions of hypotheses 1 and 2 are true for groups of the type D_4, G_2, F_4, E_6, E_7.

Results of a classification kind are also obtained during the proof of this theorem. It moreover turns out to be true that

PROPOSITION 2. If the group contains semisimple subgroups under the conditions of the theorem, then hypothesis 3 is true for G.

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In conclusion, let us note that our hypotheses are generalizations of the Serre hypotheses [3]. Namely, there holds

PROPOSITION 3. Let each algebra with division over the field k be commutative. If B is a Borel group in G, then $\mathrm{ind}_k G/B = 1$ (compare with [5], n° 11.1).

Let us assume $p \mid \mathrm{ind}_k G/B$, p a prime. If K is a decomposition field of the k-torus $T \subset G$, then $(G/B)_k \neq \emptyset$, and denoting the Sylow subgroup in $\Gamma(K/k)$ by Γ_p, we have $p \mid \mathrm{ind}_L G/B$, where $L = \{\lambda \in K : \lambda^\sigma = \lambda \,\forall \sigma \in \Gamma_p\}$.

It has been verified that for $p \neq 2$ the group G contains radical (with respect to T) anisotropic k-subgroups of type A_{p-1}. As is known, such subgroups correspond to noncommutative algebras with division over k. If $p = 2$, we arrive at a contradiction by the same method.

PROPOSITION 4. Let a complete field k possess the property: if H is a coherent simply connected k-group of type A_n, then $H^1(k, H) = 0$. Then if the group G is defined entirely from coherent simply connected quasidecomposable groups of the same type as G, then $\mathrm{ind}_k G/B = 1$.

The proof of this assertion consists of the successive application of the proof of Proposition 3 and the Harder result (see [2], n° 2.3.5).

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