INFINITE-DIMENSIONAL FILTERED LIE ALGEBRAS AND THEIR CONNECTION WITH GRADED LIE ALGEBRAS

B. Yu. Veisfeiler

A Lie algebra L over a field of characteristic O with a decreasing sequence of subspaces $L = L_{-k} \supset L_{-k+1} \supset \ldots$ will be called filtered if the following conditions are satisfied: 0) L_0 is a maximal subalgebra in L; 1) $\cap L_i = 0$; 2) $\dim L_i/L_{i+1} < \infty$; 3) $[L_i, L_j] \subset L_{i+j}$; 4) $\forall x \in L_i \setminus L_{i+1}, i \geqslant 0, \exists y \in L_{-1} : [x, y] \notin L_i$. To every filtered Lie algebra a graded Lie algebra is associated $\mathfrak{g} = \sum_{-k}^{\infty} \mathfrak{g}_i, \mathfrak{g}_i = L_i/L_{i+1}$ by the ordinary method [2]. The number k is called depth of the algebras L and $\mathfrak{g}$.. To filtered Lie algebras of depth 1 there leads the classification problem of the infinite primitive transformation pseudogroups (see [2]). This problem was considered by E. Cartan [4]. Its modern solution is contained in [3]. Until very recently, the algebraic study of filtered Lie algebras was carried out under the assumption that their depth is one and the representation L_0 onto L/L_0 is irreducible ([2] and others). The recent results of V. Kats [1] together with the considerations sketched here give a simple classification of arbitrary filtered Lie algebras.

In the present note,

1) For a given filtered algebra $L = L_{-1} \supset L_0 \supset \ldots$ of depth 1 a filtered algebra $U = U_{-k} \supset U_{-k+1} \supset \ldots$, is constructed, such that $U_{-k} = L$, $U_0 = L_0$ and conditions of "not being condensed" are satisfied: 5) $U_{i-1} = [U_i, U_{-1}]$ $\forall i < 0$ and 6) U_0/U_1 is represented faithfully and irreducible onto U_{-1}/U_0.

2) It is shown that the associated graded algebra $\mathfrak{g} = \sum \mathfrak{g}_i, \mathfrak{g}_i = U_i/U_{i+1}$ is simple (and, consequently, the classification results of [1] are applicable to it).

3) It is shown that the completion of the algebra L in the topology of the subspaces $\{U_i\}$ is isomorphic to the complete graded algebra $\bar{\mathfrak{g}} = \overline{\sum} \mathfrak{g}_i$ (the line denotes the complete sum).

Let $L = L_{-1} \supset L_0 \supset \ldots$ be a filtered Lie algebra of depth 1, $\psi : L \to L/L_0 = S$ the natural projection, S_{-1} an irreducible subspace for L_0 onto S, and T the sum of all irreducible subspaces for L_0 onto S,

$$U_0 = L_0,\ U_{-1} = \psi^{-1}(S_{-1}),\ U_{i-1} = [U_i, U_{-1}]\ \forall i < 0,\ U_{i+1} = \{x \in U_i : [x, U_{-1}] \subset U_i\}\ \forall i \geqslant 0.$$

By condition 0) the filtration $\{U_i\}$ is finite on the negative side. Inclusions 3) follow from the Jacobi identity by the usual method. The properties 4), 5), and 6) are satisfied by construction, 0) 1), and 2) follow from the corresponding properties of the algebra $L = \{L_i\}$.

The proof that $\mathfrak{g}$ is a simple algebra is carried out according to the following scheme: in the algebra $\mathfrak{g}$ an arbitrary subalgebra is chosen $\mathfrak{g}' = \sum \mathfrak{g}_i'$, such that $\mathfrak{g}_i' = \mathfrak{g}_i \forall i \leqslant 0$, $\mathfrak{g}_1'$ is irreducible with respect to $\mathfrak{g}_0$ and $[\mathfrak{g}_i', \mathfrak{g}_1'] = \mathfrak{g}_{i+1}' \forall i > 0$. We consider the ideal $\mathfrak{m} = \sum \mathfrak{m}^{(i)}$, where $\mathfrak{m}^{(0)} = 0$ and $\mathfrak{m}^{(i+1)} = \left\{x \in \sum_{i<-1} \mathfrak{g}_i : [x, \mathfrak{g}_1'] \subset \mathfrak{m}^{(i)}\right\}$. It is easy to see that the algebra $\mathfrak{g}'/\mathfrak{m}$ is simple, graded and of depth 1. Hence, in particular, it follows that $[\mathfrak{g}_{-2}, \mathfrak{g}_1] = 0$. To $\mathfrak{g}'/\mathfrak{m}$ the classification [1], [2] is applicable, from which it follows that $T = \{s \in S : [U_1, s] = 0\}$ and that all representations of $\mathfrak{g}_0$ onto T are equivalent to one and the same representation π. Since $[\mathfrak{g}_{-2}, \mathfrak{g}_1] = 0$, we have $\psi^{-1}(T) \supset U_{-2}$. In particular, all representations $\mathfrak{g}_0$ onto $\mathfrak{g}_{-2}$ are equivalent to π. Since $\mathfrak{g}_{-2} = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]$, we have $\pi \subset \pi \otimes \pi$, which is impossible. Therefore,

Institute of Theoretical and Experimental Physics. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 2, No. 1, pp. 94-95, January-March, 1968. Original article submitted September 25, 1967.

$m = 0$. Hence, it follows that the algebra $\mathfrak{g}$ is simple, since we can show that every ideal of the algebra $\mathfrak{g}$ is contained in $\sum_{i<0} \mathfrak{g}_i$. Apart from the algebra $\mathfrak{g}$ being simple, for the applicability of V. Kats's results it is necessary that $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ generate $\mathfrak{g}$. This can be shown using, for instance, a lemma analogous to that of Kobayashi-Nagano [2].

The proof of the isomorphism of the algebras L and $\tilde{\mathfrak{g}}$ is based on the following simple property of the algebra $\mathfrak{g}$: a) the representations of $\mathfrak{g}_0$ onto $\mathfrak{g}_i$ are non-equivalent for $i \neq j$. In each of the algebras U_0/U_i we can choose a subalgebra $\mathfrak{g}_0^{(i)}$, even a compatible one. The limiting algebra $\tilde{\mathfrak{g}_0}$ in the topology $\{U_i\}$ can be taken for $\mathfrak{g}_0 \subset L_0$. Analogously, using a), for an arbitrary $s \neq 0$ we can choose compatible subspaces $\mathfrak{g}_s^{(i)}$ and U_s/U_i, $i > s$.

The author renders his gratitude to É. B. Vinberg for the attention he devoted to the work.

LITERATURE CITED

1. V. G. Kats, Funktsional'nyi Analiz i Ego Prilozheniya, 1, No. 4, 82-83 (1967).
2. S. Kobayashi and I. Nagano, J. Math. Mech., 13, 875-907 (1964); 14, 513-521, 679-706 (1965); 15, 163-175, 315-329 (1966).
3. I. M. Singer and S. Sternberg, J. d'Analyse math., 15, 1-114 (1965).
4. É. Carter, Ann. Sci. École Norm. Sup., 26, 93-161 (1909).