REMARKS ON SOME ALGEBRAIC GROUPS

B. Yu. Veisfeiler

Let \( k \) be a field, \( \text{char} k \neq 2 \), \( K \) a quadratic extension of the field, and \( G \) a semisimple algebraic group which is defined over \( k \) and decomposable over \( K \). A maximum torus \( T \subset G \) is said to be allowed if it is defined over \( k \), anisotropic over \( k \), and decomposable over \( K \). Let \( \Sigma \) be a system of roots in \( G \) relative to \( T \); if \( \alpha \in \Sigma \), we shall denote by \( G_\alpha \) a three-dimensional simple subgroup generated by the root subgroups \( N_\alpha \) and \( N_{-\alpha} \), and normalizable by the torus \( T \). The Galois group \( \Gamma(K/k) =\{1, \sigma\} \) acts on the group of characters \( X(T) \) of the allowed torus \( T \).

**Lemma.** Let \( T \) be an allowed torus in \( G \). Hence a) \( \sigma \alpha = -\alpha, \forall \alpha \in X(T) \); b) the subgroups \( G_\alpha, \alpha \in \Sigma \) are defined over \( k \).

To the subgroups \( G_\alpha \) it is possible to assign central quaternion algebras \( D_\alpha \). We shall denote by \( \text{Nrd} \) a homomorphism of a reduced norm of the algebra \( D_\alpha \) into its center. The algebras \( D_\alpha \) are cyclic algebras; \( D_\alpha = (K, \lambda_\alpha), \lambda_\alpha \in k^* \text{mod } NK/k(K^*) \); in this case we shall say that the group \( G \) represents an array \( \{\lambda_\alpha\}_{\alpha \in \Sigma} \) with respect to the torus \( T \). The principal question of interest to us is as follows: How is it possible to distinguish arrays that are represented by the group \( G \) with respect to two distinct allowed tori?

An allowed torus \( T' \) is said to be associated with respect to \( \alpha \in \Sigma \) to the torus \( T \) if \( T' \subset G_\alpha T \), where \( \alpha \in \Sigma, \text{rg}_K G_\alpha = 0 \). It is easy to prove the following result:

**Theorem.** If \( \text{rg}_K G = 0 \), the group \( G \) will contain allowed tori and any two allowed tori can be obtained from each other by a finite number of transitions to associated tori.

By a simple analysis of type-\( A_1 \) groups it is possible to obtain

**Proposition 1.** If an allowed torus \( T' \) is associated with an allowed torus \( T \) with respect to \( \beta \in \Sigma \) and the group \( G \) represents the arrays \( \{\lambda_\alpha\} \) and \( \{\lambda'_\alpha\} \) with respect to the tori \( T \) and \( T' \), then \( \lambda_\alpha = \nu^{[\alpha, \beta]} \cdot \lambda_{\beta} \), \( \lambda_\alpha \), where \( \nu \in \text{Nrd} D_{\beta,k} \). Conversely, if \( \nu \in \text{Nrd} D_{\beta,k} \), then the arrays \( \lambda_\alpha \) and \( \nu^{[\alpha, \beta]} \cdot \lambda_{\alpha} \) will be represented by the group \( G \) with respect to associated tori; \( [\alpha, \beta] = 2(\alpha, \beta) - (\beta, \beta)^{-1} \).

We can also prove

**Proposition 2.** If \( \text{rg}_K G > 0 \) and the group \( G \) contains an allowed torus, then it will contain an allowed torus \( T \) such that \( \lambda_\alpha \in NK/k(K^*) \) for at least one \( \alpha \in \Sigma \) (i.e., the group \( G_\alpha \) is decomposable).

Over special fields it is possible to obtain with the aid of our results interesting corollaries.

**Corollary 1.** If \( \text{Nrd} D_k = k \) for any central quaternion algebra \( D/k \) and \( \text{rg}_K G = 0 \), then \( G \) will be of type \( A_1 \).

This is a direct consequence of Proposition 1.

**Corollary 2.** If \( \text{Nrd} D_k = NK/k(K) \) for any algebra \( D = (K, \lambda), \lambda \in k^* \), and \( \text{rg}_K G = 0 \), then any two allowed tori will be conjugate in \( G_k \).
Indeed, by virtue of our condition the properties of associativity with respect to \( \alpha \in \Sigma \) and conjugateness in \( G_{\alpha, k} \) coincide. Our assertion follows from the theorem. Let us note that the field of real numbers satisfies our conditions; therefore, Corollary 2 is a generalization of E. Cartan's well-known theorem on the conjugateness of maximum tori in a compact Lie group.

It is also possible to prove the following simple result:

**PROPOSITION 3.** If \( T \) and \( T' \) are two allowed tori in \( G \) and the group \( G \) represents with respect to \( T \) and \( T' \) the same arrays, then the tori \( T \) and \( T' \) will be conjugate in the group \((\text{Aut} \ G)_k\).

Results, similar to those presented above, can be obtained for groups containing maximum tori that are defined over \( k \) and decomposable over a Galois extension of the Galois group \( Z_p \).

The author expresses his gratitude to E. B. Vinberg and D. A. Kazhdan for their interest and valuable remarks.