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SEMISIMPLE ALGEBRAIC GROUPS WHICH ARE SPLIT OVER A QUADRATIC EXTENSION

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Abstract. We consider algebraic groups defined over a field k and containing a maximal torus T which is defined and anisotropic over k and split over a given quadratic extension K of k. We study certain structural features of such groups, and the results obtained are used to investigate the behavior of these groups over special fields.

1. **Introduction.** Borel, Tits, Satake and others have studied the structure of isotropic semisimple groups over arbitrary fields. In this present paper we consider the simplest case of anisotropic groups, namely, groups which are split over a quadratic extension of the base field. We obtain some elementary structural results for these groups (§§7, 8).

The investigation is carried out as follows. If a group G is anisotropic, it contains admissible tori, i.e. maximal tori which are defined over k and split over a given quadratic extension K. The three-dimensional subgroups generated by the vector root subgroups N_α and $N_{-\alpha}$ are defined over k and correspond to the quaternion algebras $\mathfrak{D}_\alpha = (K, \lambda_\alpha)$. We study (§8) how the λ_α depend on the admissible torus, and from the result obtained we deduce some basic consequences (§§9–17). When $k = R$ is the field of real numbers, we obtain (§11) as a corollary of our results the well-known theorem on the conjugacy of the maximal tori in a compact Lie group. If the field k satisfies Serre's condition (C_2') [2], then a corollary of our results is a special case (§12) of a conjecture of Serre ([2], §3.1).

2. **Notation and conventions.** Throughout this paper we assume that k is the base field, K a separable quadratic extension of k, $K = k(b)$, $\sigma \in \Gamma(K/k)$, $b^\sigma = 1 - b$. Let G be a semisimple algebraic group defined over k. Maximal tori of G which are defined and anisotropic over k and split over K will be called *admissible*. A group G containing an admissible torus will also be called *admissible*. If $\mathfrak{D}$ is a central associative algebra over k, the reduced norm homomorphism will be denoted by Nrd (recall that

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over $\bar{k}$, Nrd coincides with the determinant). If Σ' is a root subsystem of a root system Σ of G, we denote by $G(\Sigma')$ the algebraic subgroup of G realizing the subsystem Σ'; following E. B. Dynkin, we call such subgroups regular.

3. Suppose the group G contains an admissible torus T, B is a Borel group of G defined over K and containing T, and Σ is the root system of G relative to T. For each $\alpha \in \Sigma$ we denote by $u_\alpha: \bar{k} \longrightarrow G$ the homomorphism defined over K which imbeds the additive group of the field $\bar{k}$ into G as a root subgroup. Let G_α denote the simple three-dimensional subgroup generated by the groups $u_\alpha(\bar{k})$ and $u_{-\alpha}(\bar{k})$. Let $X(T)$ be the character group of T. Also, let $u_\alpha(1)$ be the standard generators of G (the images of the Chevalley basis under the mapping exp) (see [11], §§4.3 and 4.4).

Lemma. (a) $\sigma = 1$ *on* $X(T)$.

(b) $T = B \cap B^\sigma$.

(c) $(u_\alpha(t))^\sigma = u_{-\alpha}(\xi_\alpha t^\sigma) \forall t \in K$, *where* $\xi_\alpha \in K$.

(d) *All of the groups* G_α *are defined over* k.

(e) *If* G *is* k*-simple, then it is absolutely simple.*

(f) *If* $\pi: G \longrightarrow G'$ *is a central* k*-isogeny, then* $(\pi(u_\alpha(t)))^\sigma = \pi(u_{-\alpha}(\xi_\alpha t^\sigma))$.

Proof. By hypothesis, the torus T has no characters defined over k. Since $\sigma^2 = 1$, then $\sigma = 1$ on $X(T)$. Properties (b)–(e) follow from this without difficulty [5]. In view of (c) and (d) it is sufficient to prove (f) for groups of type A_1; in this case it is obvious.

4. **Lemma (converse to Lemma 3).** *If* $\mathrm{rk}_k G = 0$ *and* B *is any Borel* K*-group in* G, *then* $T = B \cap B^\sigma$ *is an admissible torus.*

Proof. Since $\sigma^2 = 1$ and K/k is separable, the group $T = B \cap B^\sigma$ is defined over k. Being the intersection of two Borel groups, it contains a maximal torus ([3], §2.16) and, since G is anisotropic, must coincide with it ([6], §1.1).

5. **Groups of type A_1.** Suppose that G is a simply connected admissible group of type A_1. As is well known [7], G can be identified with the kernel $\mathfrak{D}^0$ of the reduced norm homomorphism Nrd: $\mathfrak{D} \longrightarrow \bar{k}$ of a suitable quaternion algebra $\mathfrak{D}$. We use the notation of §3 and we take $\Sigma = \{\pm \alpha\}$, B the upper triangular matrices in G, B^σ the lower, T the diagonal matrices. Put $e_1 = E_{11}$, $e_2 = E_{12}$, $e_3 = E_{21}$, $e_4 = E_{22}$, $u_\alpha = E + E_{12}$ and $u_{-\alpha} = E + E_{21}$. In view if §3 we have

$$e_2^\sigma = \xi_\alpha e_2, \quad e_3^\sigma = \xi_{-\alpha} e_2, \quad e_1^\sigma = e_4, \quad e_4^\sigma = e_1.$$

Using the condition $\sigma^2 = 1$, we obtain

$$(e_2)^{\sigma^2} = e_2 = \xi_\alpha^\sigma \cdot \xi_{-\alpha}^\sigma e_2,$$

i.e. $\xi_\alpha^\sigma \cdot \xi_{-\alpha} = 1$, and

$$(e_1 - e_4) = (e_1 - e_4)^\sigma = ([e_2, e_3])^\sigma = [\xi_\alpha e_3, \xi_{-\alpha} e_2] = - \xi_\alpha \cdot \xi_{-\alpha} (e_1 - e_4),$$

i.e. $\xi_\alpha \cdot \xi_{-\alpha} = 1$. Hence $\xi_{-\alpha} = \xi_\alpha^{-1}$ and $\xi_\alpha = \xi_\alpha^\sigma$; that is, $\xi_\alpha \in k^*$.

On the other hand, according to the theory of cyclic algebras, $\mathfrak{D}_k$ is isomorphic to the algebra $K + uK$, where $u^2 = a \in k^*$, $\lambda u = u\lambda^\sigma \forall \lambda \in K$. Here a is determined by $\mathfrak{D}$ modulo the norms of K^* in k^*, i.e. $a \in k^* \bmod N(K^*)$.

The matrix $\lambda + u\mu$ ($\lambda, \mu \in K$) has, relative to the basis $\{1, u\}$, the form

$$A_{\lambda,\mu} = \lambda e_1 + \mu e_2 + a\mu^\sigma e_3 + \lambda^\sigma e_4.$$

The determinant of this matrix is equal to $\lambda\lambda^\sigma - a\mu\mu^\sigma$, i.e. the group $\operatorname{Nrd}\mathfrak{D}_k$ coincides with the group of numbers $\lambda\lambda^\sigma - a\mu\mu^\sigma$. Considering the same matrix $A_{\lambda,\mu}$ and using the theorem which states that the determinant of a product of matrices is equal to the product of the determinants, we see that the group $\mathfrak{D}^0$ preserves the Hermitian form $f(x, y) = xx^\sigma - ayy^\sigma$.

The matrices $A_1 = e_2 + ae_3$ and $A_2 = be_2 + ab^\sigma e_3$ belong to $\mathfrak{D}_k$. We have:

$$e_2 = (b^\sigma A_1 - A_2)(b^\sigma - b)^{-1}, \quad e_2^\sigma = (bA_1 - A_2) \cdot (-b^\sigma + b)^{-1} = ae_3$$

(since $b^\sigma = 1 - b$). It follows that $\xi_\alpha = a$ and that ξ_α is determined modulo $N(K^*)$.

Thus we have the following

Lemma. *Suppose that $G = \mathfrak{D}^0$, where $\mathfrak{D}$ is a central quaternion algebra which is decomposable over K. Then the following assertions are valid.*

(a) *$\mathfrak{D}_k$ is a cyclic algebra (K, a), where $a \in k^* \bmod N(K^*)$.*

(b) *$\mathfrak{D}_k = K + uK$, where $u^2 = a$ and $\lambda u = u\lambda^\sigma \forall \lambda \in K$.*

(c) *The group $\operatorname{Nrd}\mathfrak{D}_k$ coincides with the group of numbers $\lambda\lambda^\sigma - a\mu\mu^\sigma$, $\lambda, \mu \in K$.*

(d) *$\mathfrak{D}^0$ is isomorphic to the group $SU(f)$, where f is the Hermitian form $xx^\sigma - ayy^\sigma$.*

(e) *$\xi_\alpha = a$, $\xi_{-\alpha} = a^{-1}$.*

6. **Elementary study of the numbers ξ_α** We will apply the results of §3. We say that a group G *represents the set* $\{\lambda_\alpha\}_{\alpha\in\Sigma}$, $\lambda_\alpha \in k^*/N(K^*)$, *with respect to the torus* T, if $\lambda_\alpha = \xi_\alpha \cdot N(K^*)$.

The group G *represents the set* $\{\lambda_\alpha\}_{\alpha\in\Sigma}$ if a suitable admissible torus T can be found. The sets $\{\lambda_\alpha\}_{\alpha\in\Sigma}$ are obviously determined up to an automorphism of the root system, i.e. $\{\lambda_\alpha\} \cong \{\lambda_{\omega\alpha}\}$ if $\omega \in \operatorname{Aut}\Sigma$.

Suppose that Δ is a system of simple roots in Σ, and let $\mathfrak{D}_\alpha$ be the cyclic algebra (K, ξ_α).

Lemma. (a) *The group $\mathfrak{D}_\alpha^0$ is isogenous over k to the group G_α.*

(b) *The set $\{\xi_\alpha\}_{\alpha\in\Sigma}$ is completely determined by the set $\{\xi_\alpha\}_{\alpha\in\Delta}$.*

(c) *The set $\{\xi_\alpha\}_{\alpha\in\Sigma}$ is completely determined (up to a natural equivalence) by the set $\{\lambda_\alpha\}_{\alpha\in\Sigma}$*

Proof. The statement (a) follows from the fact that ξ_α and $\xi_{-\alpha}$ determine G_α as a k-form, and the fact that these numbers are the same for isogenous groups (see [5] and §3 (f)). The groups G_α, $\alpha \in \Delta$, generate the algebraic group G; hence the assignment of the ξ_α for $\alpha \in \Delta$ determines the action of σ on the whole group G_K; this implies (b).

To prove (c), we note that the replacement of $\{\xi_\alpha\}_{\alpha\in\Sigma}$ by $\{\xi_\alpha\cdot\nu_\alpha\cdot\nu_\alpha^\sigma\}_{\alpha\in\Sigma}$, where $\nu_\alpha\in K$, $\nu_{\alpha+\beta}=\nu_\alpha\cdot\nu_\beta$, $\nu_\alpha^{-1}=\nu_{-\alpha}$, determines the same group G (see [5], §13) and hence these two sets may be regarded as equivalent. Thus for $\alpha\in\Delta$ we can replace the ξ_α by any representatives $\mathrm{mod} N(K^*)$. To recover $\{\xi_\alpha\}_{\alpha\in\Sigma}$ from $\{\lambda_\alpha\}_{\alpha\in\Sigma}$ we proceed as follows: for $\alpha\in\Delta$ we take $\xi_\alpha\in\lambda_\alpha$ and, by means of (b), find all of the remaining ξ_α. This construction is seen to be correct by what has been said above.

7.**Associated tori.** We use the notation of §§3 and 6. Let $g\in G_K$, $B_g=gBg^{-1}$ and $\phi_g^B(T)=B_g\cap(B_g)^\sigma$. Suppose $\alpha\in\Delta$. The torus $\phi_g^B(T)$ is said to be associated with T via α if $g\in P_{\alpha,K}$ and $\mathrm{rk}_k G_\alpha=0$, where $P_\alpha=G_\alpha\cdot B$. If the tori T and T' are associated via $\beta\in\Delta$ and $\{\lambda_\alpha\}_{\alpha\in\Sigma}$ and $\{\lambda'_\alpha\}_{\alpha\in\Sigma}$ are sets represented by G with respect to T and T', then these sets are said to be *associated via* β.

The tori T and T_1 are said to be *joined* if we can pass from T to T_1 by a finite sequence of associated tori. We now put

$$\varphi_g^B(G_\beta)=(gP_\beta g^{-1})\cap(gP_\beta g^{-1})^\sigma\quad\forall\beta\in\Delta.$$

Let M^B be the set of those g for which the tori T and $\phi_g^B(T)$ are joined.

Proposition. *If* $\mathrm{rk}_k G=0$, *then any two admissible tori are joined in* $|W|$ *steps (where* $|W|$ *is the order of the Weyl group* W *of* G*). If* G *contains a regular subgroup* H *of type* A_1 *which is split over* k, *then there exists an admissible torus* T_1, *joined to* T *in a finite number of steps, and a group of type* A_1, *split over* k, *normalized by this torus.*

The proof is broken into several parts.

(a) If $\alpha\in\Delta$, $\mathrm{rk}_k G_2=0$ and $g\in P_{\alpha,K}$, then $\phi_g^B(T)$ is an admissible torus.

Let $T_\alpha=T\cap G_\alpha$, $S=T\cap Z(G_\alpha)$, $B_\alpha=B\cap G_\alpha$ and $T_1=(gB_\alpha g^{-1})\cap(gB_\alpha g^{-1})^\sigma$. In view of our hypotheses, $\phi_g^B(G_\alpha)=G_\alpha$, and therefore, by §4, T_1 is an admissible torus in G_α. Since $P_\alpha=B\cdot G_\alpha$ and $mSm^{-1}=S\,\forall m\in G_\alpha$, it follows that $S\subset B_g$. Since $T=T_\alpha\cdot S$, then, by what has been said above, $\phi_g^B(T)=T_1\cdot S$ is an admissible torus.

(b) If $g\in M^B$, $\alpha\in\Delta$, $\mathrm{rk}_k(\phi_g^B(G_\alpha))=0$ and $p\in P_{\alpha,K}$, then $gp\in M^B$.

Let $m=gpg^{-1}$. The assertion follows from the obvious equalities $\phi_m^{Bg}\cdot\phi_g^B=\phi_{mg}^B$ and $mg=gpg^{-1}\cdot g=gp$.

(c) If $\mathrm{rk}_k H=0$ for every regular subgroup H of type A_1 which is defined over k and split over K, then any two admissible tori are joined in $|W|$ steps.

It follows from (b) that $p_1\cdot p_2\cdots p_m\in M^B\,\forall p_i\in P_{\alpha(i),K}$, $\alpha(i)\in\Delta$. Since $G_K=B_K\cdot W\cdot B_K$ and since the reflections in the simple roots lie in $P_{\alpha(i)}$ and generate W, we have $G_K\subset M^B$ and not more than $|W|$ steps are required for joining.

(d) *The second assertion of our proposition is true.*

Let us assume that we have taken $|W|$ steps and have not encountered a split group. But then, as in (c), we have that $G_K\subset M^B$. Since H and all of the G_α, $\alpha\in\Delta$, are split over K and regular, there exist $g\in G_K$ and $\beta\in\Delta$ such that $H=g\cdot G_\beta g^{-1}$.

Since $g \in M^B$, we see that $\phi_g^B(T)$ is an admissible torus normalizing the group $H = \phi_g^B(G_\beta)$, which proves our proposition.

Remark. Since W is generated by the fundamental reflections in $|\Sigma^+|$ steps, $|W|$ may be replaced by $|\Sigma^+|$.

8. Interpretation of association.

Prosposition. *Suppose that the conditions of §§3 and 6 are satisfied. Let* T' *be the torus associated with the torus* T *via the root* $\beta \in \Delta$, *and* $\{\lambda_\alpha\}$ *and* $\{\lambda'_\alpha\}$ *sets represented by* G *with respect to* T *and* T'. *Then* $\lambda'_\alpha = \nu^{[\alpha,\beta]} \cdot \lambda_\alpha$, *where* $\nu \in \mathrm{Nrd}\mathfrak{D}^*_{\beta,k}$. *For any* $\nu \in \mathrm{Nrd}\,\mathfrak{D}^*_{\beta,k}$ *the sets* $\{\lambda_\alpha\}$ *and* $\{\nu^{[\alpha,\beta]} \cdot \lambda_\alpha\}$ *are associated via* β. *Here* $[\alpha,\beta] = 2(\alpha,\beta)/(\beta,\beta)$.

Proof. (a) Let $T' = \phi_g^B(T)$, $g \in P_{\beta,K}$, $T_\beta = T \cap G_\beta$ and $T'_\beta = T' \cap G_\beta$. Then there exists $h \in G_{\beta,K}$ such that $hT_\beta h^{-1} = T'_\beta$. Since T_β and T'_β are defined over k, then $h^\sigma T_\beta h^{-\sigma} = T'_\beta = hT_\beta h^{-1}$. Hence we have

$$(h^{-1}h^\sigma)\,T_\beta\,(h^{-1}h^\sigma)^{-1} = T_\beta.$$

Since $h \in G_\beta$ and G_β is defined over k, we have $h^{-1}h^\sigma \in G_\beta$. Thus $h^{-1} \cdot h^\sigma \in N_{G\beta}(T_\beta)$. Writing h in terms of the basis e_1, e_2, e_3, e_4 (§5), we obtain

$$h = \alpha e_1 + \beta e_2 + \gamma e_3 + \delta e_4, \quad \alpha, \beta, \gamma, \delta \in K, \quad \alpha\delta - \beta\gamma = 1;$$
$$h^{-1} = \delta e_1 - \beta e_2 - \gamma e_3 + \alpha e_4;$$
$$h^\sigma = \delta^\sigma e_1 + a^{-1}\gamma^\sigma e_2 + a\beta^\sigma e_3 + \alpha^\sigma e_4;$$
$$h^{-1} \cdot h^\sigma = (\delta\delta^\sigma - a\beta\beta^\sigma)\,e_1 + (a^{-1}\gamma^\sigma\delta - \beta\alpha^\sigma)\,e_2$$
$$+ (a\alpha\beta^\sigma - \gamma\delta^\sigma)\,e_3 + (\alpha\alpha^\sigma - a^{-1}\gamma\gamma^\sigma)\,e_4.$$

Since $\delta\delta^\sigma - a\beta\beta^\sigma \in \mathrm{Nrd}\mathfrak{D}_{\beta,k}$ (§5), it follows that $\delta\delta^\sigma - a\beta\beta^\sigma = 0$ if and only if $\delta = \beta = 0$, which is impossible. Since the matrices in $N_{G_\beta}(T_\beta)$ are monoidal, $h^{-1} \cdot h^\sigma \in T_\beta$, i.e. $a\alpha\beta^\sigma - y\delta^\sigma = 0$. Now suppose $\nu \in \mathrm{Nrd}\mathfrak{D}^*_{\beta,k}$. We will show that we can find α, β, y, δ, $\in K$ such that $h \in G_{\beta,K}$, $h^{-1} \cdot h^\sigma \in T_\beta$ and $\delta\delta^\sigma - a\beta\beta^\sigma = \nu$. For this it is sufficient to select δ and β arbitrarily and solve the system of linear (in α and y) equations

$$\begin{cases} \alpha\delta \quad - \gamma\beta = 1, \\ -\alpha(a\beta^\sigma) + \gamma\delta^\sigma = 0. \end{cases}$$

The determinant of this system is $\delta\delta^\sigma - a\beta\beta^\sigma = \nu \neq 0$. i.e. the system has a solution.

(b) Let us now look at how the set $\{\lambda_\alpha\}$ transforms. Since the torus hTh^{-1} is defined over k and the groups $hG_\alpha h^{-1} = G'_\alpha$ are normalized by hTh^{-1}, they are defined over k. Hence $hG_\alpha h^{-1} = \phi_g^B(G_\alpha)$. To the G'_α correspond the normed residues λ'_α, and we must find these. We have

$$u'_\alpha(t) = h \cdot u_\alpha(t) \cdot h^{-1}, \quad (u'_\alpha(t))^\sigma = h^\sigma \cdot u_{-\alpha}(\lambda_\alpha t^\sigma) \cdot h^{-\sigma}.$$

On the other hand,

$$(u'_\alpha(t))^\sigma = u'_{-\alpha}(\lambda'_\alpha t^\sigma) = h \cdot u_{-\alpha}(\lambda'_\alpha t^\sigma) \cdot h^{-1}.$$

Therefore

$$u_{-\alpha}(\lambda'_\alpha t^\sigma) = (h^{-1} \cdot h^\sigma)\, u_{-\alpha}(\lambda_\alpha t^\sigma)\, (h^{-1} \cdot h^\sigma)^{-1}.$$

Since $b^{-1} \cdot b^\sigma \in T_\beta$ and $b^{-1} \cdot b^\sigma$ has the form described above, then

$$u_{-\alpha}(\lambda'_\alpha t) = u_{-\alpha}(\nu^{[\alpha,\beta]} \cdot \lambda_\alpha t).$$

This proves our assertion.

9. Isotropic groups.

Proposition. *Suppose that* G *is an admissible group,* $\mathrm{rk}_k G > 0$. *Then* G *represents a set* $\{\lambda_\alpha\}$ *in which there is a* β *such that* $\lambda_\beta = 1$.

Proof. In view of §7, it is sufficient to find a regular k-subgroup of G which is k-isogenous to $SL(2)$. If an anisotropic kernel S of G contains a root k-subgroup H of type A_1 which is split over K, then the group $G_1 = Z(H)^0$ is split over K.

We may assume, because of §7 (c), that the admissible torus T normalizes H. Then G_1 contains the admissible torus $T_1 = T \cap G_1$. Applying this device several times, we obtain a quasi-split admissible group $\widetilde{G}$. Let T_0 be a maximal split torus in $\widetilde{G}$, $\widetilde{\Sigma}$ the system of k-roots of $\widetilde{G}$ with respect to T_0. The regular group $\widetilde{G}_\alpha$ corresponding to the k-root $\alpha \in \widetilde{\Sigma}$ is isogenous over k to one of the groups $SL(2)$ or $SU(3,f)$ (the form f represents 0) or $R_{K/k}(SL(2))$. In the first two cases $\widetilde{G}_\alpha$ contains a regular k-subgroup which is k-isogenous to $SL(2)$. If $\forall \alpha \in \widetilde{\Sigma}$ the group $\widetilde{G}_\alpha$ is isogenous over k to $R_{K/k}(SL(2))$, the system Δ of simple roots cannot be connected and hence $\widetilde{G} = R_{K/k}(G')$ does not contain admissible tori, which contradicts the hypotheses.

10. Conjugacy of admissible tori.

Proposition. *Suppose the conditions of §3 are satisfied, and let* T *and* T' *be two admissible tori in* G *such that* G *represents the same set with respect to these tori. Then there exists* $g \in (\mathrm{Aut} G_k)$ *such that* $g(T) = T'$.

Proof. The mapping $u_\alpha(t) \to u'_\alpha(t)$ defines an automorphism of G. Since $\lambda_\alpha = \lambda'_\alpha$, this automorphism is defined over k, which was to be proved.

11. Real-closed fields.

Proposition. *Suppose that* $\mathrm{char}\, k \neq 2$ *and for every quaternion algebra* $\mathfrak{D}/k$ *which is split over* K *we have* $\mathrm{Nrd}\,\mathfrak{D}_k = N(K)$ *(this is true if* k *is a real-closed field). If* $\mathrm{rk}_k G = 0$, *then all admissible tori in* G *are conjugate over* k.

Proof. (a) The proposition is true for the group $\mathfrak{D}^0$, where $\mathfrak{D}/k$ is a quaternion algebra which is split over K.

Indeed, according to [1], admissible tori are conjugate in the group $\mathfrak{D}_k^*$. The homomorphism Nrd: $\mathfrak{D}_k^* \to k^*$ carries the centralizer of T in $\mathfrak{D}_k^*$ (i.e. the field K) into the image of the whole group $\mathfrak{D}_k^*$. Hence the conjugacy classes of T in $\mathfrak{D}_k^*$ and in $\mathfrak{D}_k^0$ coincide, as required.

(b) *The proposition is true.* In view of what has been said above and §7, conjugacy over k and association are one and the same thing in simply connected groups. Hence the proposition is proved for simply connected groups (even if $\operatorname{char} k = 2$). In order to prove it in complete generality, it is sufficient to prove the analog of (a) for the adjoint group H of $\mathfrak{D}$. In this case it follows from the separability of the isogeny $\mathfrak{D}^0 \longrightarrow H$ (since $\operatorname{char} k \neq 2$).

12. **The field** (C_2').

Proposition. *Suppose that for every algebra* $\mathfrak{D} = (K, a)$ $(a \in k^*)$ *over the field* k *we have* $\operatorname{Nrd}\mathfrak{D}_k^* = k^*$. *If* $\operatorname{rk}_k G = 0$, *then* G *is of type* A_1.

Proof. This is a trival consequence of §8 and our hypothesis.

13. **Fields of characteristic** 2.

Proposition. *Suppose that* G *is an admissible group, and* $\operatorname{char} k = 2$. *Then there exists a purely inseparable extension of* k *over which* G *is quasi-split. In particular, if* k *is perfect,* G *is quasi-split over* k.

The proof follows immediately from the theory of algebras ([8], Theorem 7.21).

14. $\xi_{\alpha+\beta} = -\xi_\alpha \cdot \xi_\beta$.

Proposition. *Suppose that* $\operatorname{char} k > 3$. *Let* T *be an admissible torus in* G *and* $\{\xi_\alpha\}$ *a set represented by* G *with respect to* T. *If* $\alpha, \beta, \alpha+\beta \in \Sigma$, *then* $\xi_{\alpha+\beta} = -\xi_\alpha \cdot \xi_\beta$.

Proof. Suppose that $\mathfrak{g}$ and $\mathfrak{t}$ are Lie algebras of the groups G and T, $\mathfrak{t} \subset \mathfrak{g}$, and $\{E_\alpha, H_i\}$ is a Chevalley basis in $\mathfrak{g}$ defined over K. As in §5 we have $E_\alpha^\sigma = \xi_\alpha E_{-\alpha}$. Furthermore,

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}, \quad [E_{-\alpha}, E_{-\beta}] = N_{-\alpha,-\beta} E_{-\alpha-\beta},$$

and, as is well known, $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$. Applying σ to $[E_\alpha, E_\beta]$, we obtain

$$N_{\alpha,\beta}\xi_{\alpha+\beta}E_{-\alpha-\beta} = [\xi_\alpha E_{-\alpha}, \xi_\beta E_{-\beta}] = \xi_\alpha\xi_\beta N_{-\alpha,-\beta}E_{-\alpha-\beta}.$$

This implies the proposition, since $N_{\alpha,\beta} \neq 0$ under our hypotheses.

15. **Invariants of some admissible groups.** Suppose the simple roots are enumerated in the following way:

$$A_n \quad 1 - 2 - 3 - \dots - n$$

$$B_n \quad 1 - 2 - \dots - (n-1) \Rightarrow n$$

$$C_n \quad 1 - 2 - \dots - (n-1) \Leftarrow n$$

$$D_n \quad 1 - 2 - \dots - (n-2)\begin{array}{l} \diagup (n-1) \\ \diagdown n \end{array}$$

$$E_7 \quad \begin{array}{ccccccccccc} 1 & - & 2 & - & 3 & - & 4 & - & 5 & - & 6 \\ & & & & | & & & & & & \\ & & & & 7 & & & & & & \end{array}$$

Theorem. *Suppose that G is an admissible group,* $\mathrm{rk}_k G = 0$, *and* $\{\lambda_\alpha\}$ *a set represented by G with respect to the admissible torus T. Put* $x_i = \lambda_{\alpha_i}$ $\forall \alpha_i \in \Delta$. *Then the following expressions are invariants of G (i.e. do not depend on the choice of the admissible torus T).*

$$\begin{gathered} x_1 \cdot x_3 \ldots x_{2n-3} \cdot x_{2n-1}, \text{ if } G \text{ is of type } A_{2n-1}, \\ x_n \cdot x_{n-1}, \text{ if } G \text{ is of type } D_n, \\ x_1 \cdot x_3 \ldots x_{2n-1}, x_{2n-1} \cdot x_{2n}, \text{ if } G \text{ is of type } D_{2n}, \\ x_1 \cdot x_3 \ldots x_{2n-1}, \text{ if } G \text{ is of type } B_{2n} \text{ or } B_{2n-1}, \\ x_n, \text{ if } G \text{ is of type } C_n, \\ x_4 \cdot x_6 \cdot x_7, \text{ if } G \text{ is of type } E_7. \end{gathered}$$

All of these invariants belong to the group $k^*/N(K^*)$.

Proof. In §7 we showed that successive passages to associated tori via simple roots allow us to join any two given admissible tori in a finite number of steps. Hence we need only show that the above expressions are preserved under passage to associated sets (via $\beta \in \Delta$). We have $\beta = \alpha_s$. Consequently

$$x_i' = x_i \cdot \nu^{[\alpha_i, \alpha_s]}, \quad \nu \in \mathrm{Nrd}\, \mathfrak{D}^*_{\beta,k}.$$

Let I denote that set of indices for which we want to prove the invariance of the expression $\Pi_{i \in I} x_i$. If $s \in I$, then $x_i' = x_i$ for $i \in I$, since association affects only adjacent roots, and roots connected to the α_i, $i \in I$, do not belong to I.

If $s \notin I$, there are three possibilities: a) α_s has two neighbors with subscripts in I; in this case, as is evident from the Dynkin diagram, $x_i' = x_i \forall i \neq s+1, s-1$, $x'_{s+1} = x_{s+1} \cdot \nu$ and $x'_{s-1} = x_{s-1} \cdot \nu$, i.e. the invariant is multiplied by $\nu^2 \in N(K^*)$; b) α_s has no neighbors with subscripts in I; in this case association does not affect the roots with subscripts in I, i.e. the invariant is preserved; c) α_s has one neighbor, say α_k, with a subscript in I; this happens when G is of type B_{2n}, $s = 2n$, $k = 2n-1$ and when G is of type C_n, $s = n-1$, $k = n$; in these cases we have $x_k' = x_k \cdot \nu^{[\alpha_k, \alpha_s]}$, and $[\alpha_k, \alpha_s] = 2$, i.e. the invariant is multiplied by $\nu^2 \in N(K^*)$.

Remark. In the case $G = SU(f)$, our invariant coincides up to sign with the discriminant of the form f.

16. Elements of the form $g^{-1}g^\sigma$.

Theorem. *Suppose that G is a simply connected admissible group, T an admissible torus in G, and* $\mathrm{rk}_k G = 0$. *If* $g \in G_K$ *and* $g^{-1}g^\sigma \in N(T)$, *then* $g^{-1}g^\sigma \in T_r$ *(where* $T_r = \{t \in T_K : t^\sigma = t^{-1}\} = \{t \in T : \alpha(t) \in k\ \forall \alpha \in \Sigma\}$). *Let* $V = \{g \in G_K : g^{-1}g^\sigma \in T\}$. *Then* $G_K = V \cdot U_K$ *(where U is the unipotent part of a Borel group $B \supset T$). In particular,*

$$\{g^{-1}g^\sigma, g \in G_K\} = \{u^{-1}tu^\sigma, u \in U_K, t \in T_r\}.$$

Proof. Let $\{\lambda_\alpha\}$ be a set represented by G with respect to T.

a) If $h \in V$, then $h^{-1}h^{\sigma} \in T_r$.

Indeed, $(h^{-1}h^{\sigma})^{\sigma} = h^{-\sigma}h = (h^{-1}h^{\sigma})^{-1}$.

b) If $h \in G_K$ and $h^{-1}h^{\sigma} \in N(T)$, then $T' = hTh^{-1}$ is an admissible torus. If $h \in V$, then $\lambda'_{\alpha} = \lambda_{\alpha} \cdot \alpha(t)$, where $t = h^{-1}h^{\sigma}$.

Indeed, the torus T' is defined over k, since $h^{-1}h^{\sigma} \in N(T)$; it is split over K since it is conjugate over K to a torus which is split over K; it is anisotropic since G is anisotropic. The proof of the second assertion coincides word for word with the argument in §8b.

c) If $h \in V$ and $n \in N(T)_K$, then $hn \in V$.

Indeed, $(hn)^{-1} \cdot (hn)^{\sigma} = n^{-1}tn^{\sigma}$, where $t = h^{-1}h^{\sigma} \in T$. Since $\sigma\alpha = -\alpha \ \forall \alpha \in \Sigma$, then σ lies in the center of the Weyl group and hence $n^{\sigma} \in nT$. This implies our assertion.

d) If $h^{-1}h^{\sigma} \in N(T)$, then $h \in V$.

Suppose $h \in V$, $h^{-1}h^{\sigma} = t \in T_r$ and $T' = hTh^{-1}$. The groups $G'_{\alpha} = hG_{\alpha}h^{-1}$ are defined over k, since they are normalized by T. Let T'' be the torus associated with the torus T' via $\beta \in \Sigma$. As we have already observed in §8a, $T'' = gT'g^{-1}$, where $g \in G'_{\beta,K}$ and $g^{-1}g^{\sigma} \in T' \cap G'_{\beta}$. We have $T'' = (gh)T(gh)^{-1}$ and hence $(gh)^{-1}(gh)^{\sigma} \in N(T)$. Put $g = hmh^{-1}$, $m \in G_{\beta,K}$. Then $gh = hm$ and we must consider $(hm)^{-1}(hm)^{\sigma}$. We have $(hm)^{-1}(hm)^{\sigma} = m^{-1}tm^{\sigma} \in N(T)$. As in §8a,

$$m = \alpha e_1 + \beta e_2 + \gamma e_3 + \delta e_4,$$

$$m^{\sigma} = \delta^{\sigma} e_1 + a\gamma^{\sigma} e_2 + a^{-1}\beta^{\sigma} e_3 + \alpha^{\sigma} e_4,$$

$$m^{-1} = \delta e_1 - \beta e_2 - \gamma e_3 + \alpha e_4,$$

$$t = \lambda e_1 + \mu e_2,$$

$$m^{-1}tm^{\sigma} = (\lambda\delta\delta^{\sigma} - \mu a\beta\beta^{\sigma})\, e_1 + \dots .$$

We will assume that $m^{-1}tm \notin T$ and will obtain a contradiction. Indeed, in this case

$$\lambda\delta\delta^{\sigma} - \mu a\beta\beta^{\sigma} = 0,$$

which means that $a \cdot \mu/\lambda \in N(K^*)$. In view of b), we have

$$\lambda'_{\beta} = a \cdot \alpha(t) = a \cdot \frac{\mu}{\lambda} \in N(K^*),$$

which means that $\mathrm{rk}_k G'_{\beta} = 1$, i.e. $\mathrm{rk}_k G > 0$, a contradiction. Hence $m^{-1}tm^{\sigma} \in T$.

Thus we have proved the following: if an admissible torus T'' is joined to T and $g_1 T g_1^{-1} = T''$ (in our case $g_1 = gh = hm$), then $g_1^{-1}g_1^{\sigma} \in T$. But any two admissible tori are joined (§7) and hence for any admissible torus T we can say that if $h^{-1}Th = \tilde{T}$, then $h^{-1}h^{\sigma} \in T$.

To prove our assertion, we now need only quote b).

e) If B' is any Borel K-group in G, then $\exists\, h \in V$: $hBh^{-1} = B'$. In particular, $G_K = V \cdot U_K$.

Indeed, let $T' = B' \cap B'^{\sigma}$. Then $\exists b_1 \in V: b_1 T b_1^{-1} = T'$. Let $B_1 = b_1^{-1} B b_1$. The group B_1 contains the torus T and hence $B_1 = nBn^{-1}$, $n \in N(T)_K$. Put $b = b_1 \cdot n$. Then $b_1 \cdot n \in V$ (see c)) and $bBb^{-1} = B'$, as required.

Remark. If G is isotropic, our assertion is invalid.

Remark. If $G = SU(f)$ and $f = \Sigma \lambda_i x_i x_i^{\sigma}$, then

$$\{A^{-1}A^{\sigma},\ A \in G_K\} = \{AA^*,\ A \in G_K\}.$$

In particular, if $AA^* \in T_K$, then

$$A \cdot A^* = \operatorname{diag}(\lambda_1^{-1} f(e_1),\ \lambda_2^{-1} \cdot f(e_2),\ \dots,\ \lambda_n^{-1} \cdot f(e_n)),$$

where $e_1, e_2, \cdots, e_n$ is an orthogonal basis of the underlying space.

Remark. If $\phi_1, \phi_2, \cdots, \phi_r$ are the characters of the fundamental representations of G, then the sets

$$X_i = \{\varphi_i(t),\ t = h^{-1}h^{\sigma} \in T,\ h \in G_K\}, \quad i = 1, 2, \dots, r = \operatorname{rk} G,$$

are invariants of G. They clearly do not depend on the choice of the admissible torus T.

17. Inseparable extensions.

Theorem. *Suppose that* char $k = p \neq 2$, *L is a purely inseparable extension of k, G is an admissible k-group, and T is an admissible k-torus in G. If T'_α is an admissible L-torus joined to T over L, and G represents the set $\{\lambda_\alpha\}$ with respect to T' (over L), then G represents the set $\{N_{L/k}(\lambda_\alpha)\}$ over k and this set is representable with respect to a k-torus which is joined to T (over k).*

Corollaries. a) *If* $\operatorname{rk}_L G > 0$, then $\operatorname{rk}_k G > 0$.

b) *If G_1 and G_2 are two admissible k-groups and G_1 is isomorphic to G_2 over L, then G_1 is isomorphic to G_2 over k.*

c) *If G_1 is an admissible L-group, there exists a k-group G such that $G_1 \underset{L}{\simeq} G$.*

Proof of the corollaries. a) If $\operatorname{rk}_L G > 0$, then (§9) G represents a set $\{\lambda_\alpha\}$ over L with respect to a torus joined to T over L, with $\lambda_\gamma = 1$ for a suitable γ. By the theorem, G represents the set $\{\lambda'_\alpha = N_{L/k}(\lambda_\alpha)\}$ over k. Obviously $\lambda'_\gamma = 1$, i.e. $\operatorname{rk}_k G > 0$.

b) If $G_1 \underset{L}{\simeq} G_2$, then G_1 and G_2 represent the same set over L. We may assume that $\operatorname{rk}_k G_i = 0$ ($i = 1, 2$) (otherwise we consider the centralizer of a maxiaml k-trivial subtorus). According to the theorem and §7, our groups represent the same set over k, i.e. they are isomorphic over k.

c) Suppose G_1 represents the set $\{\lambda_\alpha\}$ over L. We construct an admissible k-group G with respect to the set $\{N_{L/k}(\lambda_\alpha)\}$ (§6). Since $N_{L/k}(\lambda_\alpha) = \lambda_\alpha^q$, $q = p^r = [L:k]$, it follows that $N_{L/k}(\lambda_\alpha) \equiv \lambda_\alpha \bmod N_{LK/L}(KL)$, i.e. G and G_1 represent the same set over

L, i.e. they are isomorphic over L.

The proof of the theorem is based on a theorem of Albert ([2], §2.2) and on the following lemma.

Lemma. *Suppose that* $\mathfrak{D}$ *is a central simple algebra over the field* k, *and* L *an extension of* k. *Then*

$$N_{L/k}(\mathrm{Nrd}\,\mathfrak{D}_L) \subset \mathrm{Nrd}\,\mathfrak{D}_k.$$

Proof. We have $\mathfrak{D}_L \subset \mathfrak{M}_{q,k} \otimes \mathfrak{D}_k = \mathfrak{A}$, where $q = [L: k]$. We denote by det the reduced norm homomorphism of $\mathfrak{A}$ into k. Then $\det \mathfrak{A} \subset \mathrm{Nrd}\mathfrak{D}_k$. On the other hand, when $v \in \mathfrak{D}_L$ we clearly have

$$\det v = N_{L(v)/k}(v) = N_{L/k}(N_{L(v)/L}(v)).$$

This implies our assertion.

We now prove the theorem. We have $[L: k] = q = p^r$, $p \neq 2$, and $N_{L/k}(a) = a^q \ \forall a \in L$. Let T be an admissible k-torus in G, $\{\lambda_\alpha\}$ a set represented by G with respect to T. Since $2 \nmid q$,

$$\lambda_\alpha^q \equiv \lambda_\alpha \bmod N_{LK/L}((KL^*)),$$

i.e. our assertion is true for this set. Let us assume our assertion is true for some set $\{\lambda_\alpha\}$. We will show that it is true for a set associated with it over L. Let $\beta \in \Sigma$ and $\nu \in \mathrm{Nrd}\,\mathfrak{D}_{\beta,L}$. By a theorem of Albert ([2], §2.2), the algebra $\mathfrak{D}_\beta$ is defined over k and, by the lemma, $N_{L/k}(\nu) = \nu_1 \in \mathrm{Nrd}\,\mathfrak{D}_{\beta,k}$. We have $\{N_{L/k}(\lambda_\alpha \nu^{[\alpha,\beta]})\} = \{\lambda_\alpha \nu_1^{[\alpha,\beta]}\}$, and our theorem now follows from §§7 and 8.

18. Appendix. Semisimple algebraic groups containing a maximal torus which splits over a simple cyclic Galois extension.

A0. In this appendix, some of the results relating to quadratic extensions are extended to extensions of prime degree $p > 2$.

A1. **Notation and conventions.** We assume throughout that k is the base field, p a fixed prime, K a Galois extension of k with Galois group Z_p, $\Gamma = \Gamma(K/k)$ ($\cong Z_p$), $\sigma \in \Gamma$, and $N(K^*)$ the group of norms of K^* in k^*. We assume that $K = k(b)$, where b, $b^\sigma, \cdots, b^{\sigma^{p-1}}$ is a basis for K/k; as is well known, $\det((\sigma^i\sigma^j)b) \neq 0$. Let G be a semisimple algebraic group defined over k. A maximal torus T of G which is defined and anisotropic over k and splits over K will be called *admissible*. If G contains an admissible torus, it will also be called *admissible*. Let Σ be the root system of G relative to T.

We first note that the admissiblility of G implies the existence in the group $\mathrm{Aut}\,\Sigma$ of an element τ of order p having no fixed points in the space $\Sigma \cdot Q$. If Σ is connected, $\mathrm{Aut}\,\Sigma$ contains such an element τ only in the following cases ($p > 2$):

$$A_{p-1} \text{ for any } p;$$

$$G_2, D_4, F_4, E_6, E_8 \text{ for } p = 3;$$

$$E_8 \text{ for } p = 5.$$

We also note that the class of admissible groups is not empty. This class contains the groups of type D_4, F_4 and E_6 ($p = 3$) related to the Jordan division algebra constructed by Albert [9].

A 2. Suppose the group G contains an admissible torus T, B is a Borel group in G defined over K and containing T, Σ is the root system of G relative to T, and $u_\alpha(k)$ and $u_\alpha(1)$ are chosen as in §3. If Σ' is a root subsystem of Σ, then $G(\Sigma')$ denotes the algebraic subgroup of G generated by the groups $u_\alpha(k)$, $\alpha \in \Sigma'$. If $\alpha \in \Sigma$, then Σ_α is the subsystem of Σ generated by the roots $\alpha, \alpha^\sigma, \cdots, \alpha^{\sigma p-1}$. Put $G^\alpha = G(\Sigma_\alpha)$.

Lemma. (a) *σ has no fixed points on $X(T)$.*

(b) $T = \bigcap_{i=0}^{p-1} B^{\sigma^i}$.

(c) $(u_\alpha(t))^\sigma = u_{\alpha^\sigma}(\xi_\alpha t^\sigma)$ $\forall t \in K$, *where* $\xi_\alpha \in K^*$.

(d) *All of the groups G^α are defined over k.*

(e) *If G is k-simple, then it is absolutely simple.*

(f) *The numbers ξ_α, $\alpha \in \Sigma$, depend only on the class of a central k-isogeny of G* (*i.e.* $(\pi(u_\alpha(t)))^\sigma = \pi(u_{\alpha^\sigma}(\xi_\alpha t^\sigma))$, *if π is a central k-isogeny*).

(g) *Σ_α is a subsystem of type A_{p-1} if Σ is connected and* $\operatorname{rk} \Sigma > 2$.

Proof. Statements (a)–(f) are proved as in §2. Property (g) is proved as follows: Σ_α is a root system of rank $(p-1)$ (since $\alpha, \alpha^\sigma, \cdots, \alpha^{\sigma p-1}$ generate $Q \cdot \Sigma_\alpha$ and, by (a), $\Sigma_{i=0}^{p-1} \alpha^{\sigma^i} = 0$). The group $\operatorname{Aut} \Sigma_\alpha$ contains an element of order p. These properties are possessed only by the systems of type A_{p-1} and, for $p = 3$, the system of type G_2. Since no system Σ of rank greater than two contains a subsystem G_2, (g) is proved.

A 3. **Lemma.** *Suppose that G is a group of type A_{p-1},* $\operatorname{rk}_k G = 0$, *and P is a maximal parabolic K-group in G corresponding to the natural representation. Then* $T = \bigcap_{i=0}^{p-1} P^{\sigma^i}$ *is an admissible torus.*

The proof of the lemma employs the same reasoning as does the proof of the lemma in §4; the nontriviality of the intersection $\bigcap_{i=0}^{p-1} P^{\sigma^i}$ is guaranteed by the theorem on the dimension of an intersection.

A4. **Groups of type A_{p-1}.** Let G be an admissible simply connected group of type A_{p-1}. As is well known [6], G can be identified with the group $\mathfrak{D}^0$ of units of some cyclic algebra [3] $\mathfrak{D} = (K, \sigma, a)$, $a \in k^*$ $\operatorname{mod} N(K^*)$. We will assume that $G = SL(p)$, with T the diagonal matrices, B the upper triangular matrices. We have $E_{i,i}^\sigma = E_{i+1,i+1}$ and $E_{i,j}^\sigma = \xi_{i,j} E_{i+1,i+1}$, where the indices are reduced mod p and $\xi_{i,j} \in K^*$.

Lemma. (a) $\mathfrak{D}_k \cong K + uK + \cdots + u^{p-1}K$, *where* $u^p = a \in k^*$ $\operatorname{mod} N(K^*)$ *and* $\lambda u = u\lambda^\sigma$ $\forall \lambda \in K$.

(b) *We may assume that*

$$E_{12}^\sigma = E_{23}, \ \ldots, \ E_{p-2,p-1}^\sigma = E_{p-1,p}, \ E_{p-1,p}^\sigma = aE_{p,1}, \ E_{p,1}^{0} = a^{-1} E_{1,2},$$

i.e. that $\xi_{\alpha_1} = \xi_{\alpha_2} = \cdots = \xi_{\alpha_{p-2}} = 1$, $\xi_{\alpha_{p-1}} = a$.

A5. **The set** $\{\lambda_\alpha\}$. According to §A2, the system Σ decomposes into a union of subsystems of type A_{p-1}.

It turns out that in Σ_α we can choose only one system of simple roots $\Delta_\alpha = \{\alpha_1, \dots, \alpha_{p-1}\}$ such that $\tau\alpha_i = \alpha_{i+1}$ $\forall i \leq p-2$, $\tau\alpha_{p-1} = -\sum_{i=0}^{p-1}\alpha_i$ and Δ_α, among all systems possessing the first property, contains the largest number of positive roots. If this is done, then to each subsystem Σ_α we can *uniquely* assign (by §A4) a normed residue $\lambda_\alpha \in k^* \bmod N(K^*)$ and an algebra $\mathfrak{D}_\alpha = (K, \sigma, a_\alpha)$, $a_\alpha \in \lambda_\alpha$. We say that G represents the set $\{\lambda_\alpha\}$ with respect to the torus T.

We say that the k-torus T' is associated with the admissible torus T via Σ_α if $T' \subset G^\alpha \cdot T$ and $\operatorname{rk}_k G^\alpha = 0$.

A6. **Interpretation of association.**

Proposition. *Suppose the conditions of §§A3 and A4 are satisfied. Let T' be a torus associated with T via Σ_β, and $\{\lambda_\alpha\}$ and $\{\lambda'_\alpha\}$ sets represented by G with respect to T and T'. Then $\lambda'_\alpha = \nu^{\{\alpha,\beta\}} \cdot \lambda_\alpha$, where $\nu \in \operatorname{Nrd}\mathfrak{D}^*_{\beta,k}$ and $\{\alpha, \beta\}$ is a number depending only on Σ_α and Σ_β. For any $\nu \in \operatorname{Nrd}\mathfrak{D}^*_{\beta,k}$ the sets $\{\lambda_\alpha\}$ and $\{\nu^{\{\alpha,\beta\}} \cdot \lambda_\alpha\}$ are associated via Σ_β.*

Proof. a) Suppose $T' \subset G^\beta \cdot T$, $T_\beta = G^\beta \cap T$ and $T'_\beta = G^\beta \cap T'$. Take $h \in G^\beta_K$: $hTh^{-1} = T'$. Then $hT_\beta h^{-1} = T'_\beta$. Since T and T' are defined over k, then $h^\sigma T_\beta h^{-\sigma} = T'_\beta$, i.e. $h^{-1}h^\sigma \in N_{G^\beta}(T_\beta)_K = N$. If $s \in N$, instead of the element h we may consider hs, which will possess all of the same properties as h. We want to show that, replacing h by hs, we can find an element $g \in G^\beta_K$ such that $gTg^{-1} = T'$ and $g^{-1} \cdot g^\sigma \in T_\beta$.

We formulate this problem for substitution groups. Let $\omega \in S_p$ (S_p is the group of permutations of p symbols) represent the element $h^{-1}h^\sigma \in N$. If $s \in N$, then $s^{-1}h^{-1}h^\sigma s^\sigma$ represents $r_1^{-1}\omega\sigma\tau_1\sigma^{-1}$ (here, of course, τ_1 represents s). The condition

$$(h^{-1}h^\sigma) \cdot (h^{-\sigma}h^{-\sigma^2}) \dots (h^{-\sigma^{p-1}}h) = 1$$

takes the form

$$\omega \cdot (\sigma\omega\sigma^{-1}) \cdot (\sigma^2\omega\sigma^{-2}) \dots (\sigma^{p-1}\omega\sigma) = 1,$$

which, after rearrangement of parentheses, becomes $(\omega\sigma)^p = 1$. We consider two cases: $\omega\sigma \neq 1$ and $\omega\sigma = 1$. If $\omega\sigma \neq 1$, then $\omega\sigma = \tau_1\sigma\tau_1^{-1}$ (since all elements of order p in the group S_p are conjugate) and hence $\tau_1^{-1}\omega\sigma\tau_1\sigma^{-1} = 1$, i.e. our assertion is true (there is an $s \in N$ such that $(hs)^{-1}(hs)^\sigma \in T$). If $\omega\sigma = 1$, then $\omega = \sigma^{p-1}$ and we analyze this case separately. So as not to have to draw very large matrices, we analyze only the case $p = 3$.

We have

$$h = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix}, \quad h^\sigma = \begin{pmatrix} \gamma_3^\sigma & a^{-1}\alpha_3^\sigma & a^{-1}\beta_3^\sigma \\ a\gamma_1^\sigma & \alpha_1^\sigma & \beta_1^\sigma \\ a\gamma_2^\sigma & \alpha_2^\sigma & \beta_2^\sigma \end{pmatrix}.$$

We assume that

$$m = h^{-1}h^{\sigma} = \begin{pmatrix} 0 & \lambda & 0 \\ 0 & 0 & \mu \\ \lambda^{-1}\mu^{-1} & 0 & 0 \end{pmatrix}.$$

From the equation $h^{\sigma} = hm$ we obtain

$$\alpha_1 = \lambda^{-1}a^{-1}\alpha_3^{\sigma}, \quad \alpha_2 = \lambda^{-1}\alpha_1^{\sigma}, \quad \alpha_3 = \lambda^{-1}\alpha_2^{\sigma}.$$

Substututing these equations into one another, we see that $\alpha_2 = \lambda^{-1}\lambda^{-\sigma}\lambda^{-\sigma^2}\alpha^{-1}\alpha_2$, i.e. $a \in N(K^*)$, a contradiction. Thus $h^{-1}h^{\sigma} \in T_{\beta}$ and therefore

$$m = h^{-1}h^{\sigma} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda^{-1}\mu^{-1} \end{pmatrix}.$$

Using again that $h^{\sigma} = h \cdot m$, we have

$$\alpha_1 = \lambda^{-1}\gamma_3^{\sigma}, \quad \alpha_2 = \lambda^{-1}a\gamma_1^{\sigma}, \quad \alpha_3 = \lambda^{-1}a\gamma_2^{\sigma},$$
$$\beta_1 = \mu^{-1}a^{-1}\alpha_3^{\sigma} = \lambda^{-\sigma}\mu^{-1}\gamma_2^{\sigma^2}, \quad \beta_2 = \mu^{-1}\alpha_1^{\sigma} = \lambda^{-\sigma}\mu^{-1}\gamma_3^{\sigma^2},$$
$$\beta_3 = \mu^{-1}\alpha_2^{\sigma} = \lambda^{-\sigma}\mu^{-1}a\gamma_1^{\sigma^2}.$$

Consequently

$$h = \begin{pmatrix} \lambda^{-1}\gamma_3^{\sigma} & \lambda^{-\sigma}\mu^{-1}\gamma_2^{\alpha^2} & \gamma_1 \\ \lambda^{-1}a\gamma_1^{\sigma} & \lambda^{-\sigma}\mu^{-1}\gamma_3^{\sigma^2} & \gamma_2 \\ \lambda^{-1}a\gamma_2^{\sigma} & \lambda^{-\sigma}\mu^{-1}a\gamma_1^{\sigma^2} & \gamma_3 \end{pmatrix}, \quad \det h = 1.$$

Put

$$h' = \begin{pmatrix} \gamma_3^{\sigma} & \gamma_2^{\sigma^2} & \gamma_1 \\ a\gamma_1^{\sigma} & \gamma_3^{\sigma^2} & \gamma_2 \\ a\gamma_2^{\sigma} & a\gamma_1^{\sigma^2} & \gamma_3 \end{pmatrix}.$$

Then $h' \in \mathfrak{D}_{\beta,k}$ and $\det h = \lambda^{-1}\lambda^{-\sigma}\mu^{-1}\det h' = 1$. But we know that $\det h' = \nu \in \mathrm{Nrd}\,\mathfrak{D}^*_{\beta,k}$; hence

$$\lambda\lambda^{\sigma}\mu = \nu^{-1} \in \mathrm{Nrd}\,\mathfrak{D}^*_{\beta,k}.$$

Conversely, if we are given ν, then, choosing λ and μ suitably (for example, $\lambda = 1$, $\mu = \nu^{-1}$), we can find a matrix h for which $h^{-1} \cdot h^{\sigma} = m$.

b) We now look at how the set $\{\lambda_{\alpha}\}$ transforms. First of all, we take $t \in T_{\beta,K}$, $t = \mathrm{diag}(\lambda^{-1}, \lambda, 1)$, and replace h by ht. We then have $h^{-1}h^{\sigma} = \mathrm{diag}(1, \pi, \pi^{-1})$. Our condition $\lambda\lambda^{\sigma}\mu \in \mathrm{Nrd}\,\mathfrak{D}^*_{\beta,k}$ now means that $\pi^{-1} \in \mathrm{Nrd}\,\mathfrak{D}^*_{\beta,k}$. Our proposition now clearly follows from the same arguments as in §7b and from §A5 (the canonical choice of the system of simple roots in Σ_{β}).

A7. **The nontriviality of the index of anisotropic admissible groups for** $p = 3$. Using §A6, we can partially simplify and make more conceptual the proof of the nontriviality of the index of anisotropic admissible groups for $p = 3$ (see [10]).

Theorem. *Suppose* L *is an extension of the field* k *of degree* m, $(3, m) = 1$. *If* G *is an admissible group,* $p = 3$ *and* $\mathrm{rk}_k G = 0$, *then* $\mathrm{rk}_L G = 0$.

Lemma. *Suppose* L *is an extension of the field* k, G *an admissible group,* $p = 3$, $\mathrm{rk}_k G = 0$, *and* T *an admissible* k-*torus in* G. *Then there exists an admissible* L-*torus* T' *in* G *associated with* T *such that* G *represents a set* $\{\lambda'_\alpha\}$ *with respect to* T' *in which* $\lambda'_\gamma = 1$ *for some* $\gamma \in \Sigma$.

From this assertion and the lemma of §17 the theorem is deduced by the same reasoning as in the proof of the theorem of §17. The lemma is actually proved in §§3.6, 4.2, 5.2 and 7.4 of [10].

19. **Concluding remarks.** The study of association was of fundamental importance in this paper. In this connection, we remark that the result of §9 extends (after the appropriate changes in terminology) to the groups which are isotropic (but, perhaps, not split) over a given quadratic extension K.

We also note the connection between the passage to associated tori and the method employed in [4] (§58.1); for unitary groups both of these methods coincide.

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