THE HASSE PRINCIPLE FOR ALGEBRAIC GROUPS
SPLIT OVER A QUADRATIC EXTENSION

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At the present time the Hasse principle is known to be valid for the majority of algebraic groups. However, its proof [1] makes use of the classification of algebraic groups. In this paper it will be shown that for groups which are split over a quadratic extension, the Hasse principle is a result of the strong approximation theorem for split groups and the Hasse principle for quaternions. We shall essentially use the approach given in [3, 4]. Let \( k \) be a field of algebraic numbers; \( K \) a quadratic extension, \( \sigma \in \Gamma (K / k) \), \( \sigma \neq 1 \), \( R \) (respectively, \( R_a \)), the set of all valuations of \( k \) (respectively, Archimedian valuations of \( k \)). Furthermore, let \( U_r \) denote the identity of the field \( k_r (r \in R) \), \( R' = \{ r \in R : K_r = k_r \oplus k_r \}, R'' = \{ r \in R : K_r \) is a field\}, \( R_0 = \{ r \in R'' : K_r / k_r \) is unramified\}. Let \( G \) be a semisimple simply connected algebraic group defined over \( k \) and split over \( K \). As in [3, 4], we call a maximal subtorus in \( G \) "admissible" if it is defined and is anisotropic over \( k \) and split over \( K \). Let \( \{ \lambda_a \}_{a \in \mathbb{Z}}, \lambda_a \equiv k^* \mod N (K^*) \), denote the set being represented by the group \( G \) with respect to the admissible torus \( T \). Let \( R (T, a) = \{ r \in R : \lambda_a \notin N (K_r) \}, R (T) = \bigcup R (T, a) \), and let \( N \) denote the norm from \( K \) to \( k \), and from \( K_r \) to \( k_r \).

Definition. Let \( G, H \) be semisimple algebraic groups over \( k \), and for each \( r \in R \) let there be given a \( k_r \)-isomorphism \( \varphi_r : G \rightarrow H \). We call the system \( \{ \varphi_r \}_{r \in R} \) "consistent" if for any class of parabolic subgroups \( \mathcal{P} \) in \( G \) and \( \mathcal{P}' \) in \( H \)

\[
\exists r \in R : \varphi_r (\mathcal{P}) = \mathcal{P}' \Rightarrow \psi_r (\mathcal{P}) = \mathcal{P}' \quad \forall r \in R.
\]

(If \( H = G \) and \( \mathcal{P} = \mathcal{P}' \), then the expression "system of consistent isomorphisms" is synonymous with the expression "system of inner automorphisms."

Let \( T_r \) denote \( k_r \)-tori in \( G \) and let \( \Delta_r = \{ a_n, \ldots, a_1 \} \) be an ordered system of simple roots in the root system of the group \( G \) with respect to \( T_r \). We say that the system \( \Delta_r \) is "consistent" if for all \( i \in [1, n] \) there exists a class \( \mathcal{P}_i \) of maximal parabolic subgroups in \( G \) such that in all \( \Delta_r \) the root \( \alpha_i, r \) corresponds to the class \( \mathcal{P}_i \).

THEOREM. Let \( R \subset R'', R \supset R (T) \cup (R'' \setminus R_a) \cup R_a, |R| < \infty \). Let the group \( G \) be anisotropic over \( k \), and let \( T_r, r \notin R, \) be an admissible \( k_r \)-torus in \( G \). Let \( \Delta_r \) be consistent systems of simple roots of \( G \) with respect to \( T_r \), and let \( \{ \lambda_a \}_{a \in \mathbb{Z}} \) be a set represented by the group \( G \) with respect to the torus \( T_r \) (over \( k_r \)). Then there exists an admissible \( k \)-torus \( T' \) and a system of simple roots \( \Delta' \) with respect to \( T' \) satisfying the conditions:

a) \( \Delta' = \{ \alpha' \}, \ldots, \alpha_n \} \) is consistent with \( \Delta_r \) for all \( r \notin R; \)

b) if \( \{ \lambda_a \}_{a \in \mathbb{Z}} \) is a set represented by the group \( G \) with respect to \( T' \), then \( \forall i \in [1, n] \) we have \( \lambda_i, r \subseteq \lambda, r \cdot N (K_r), \forall r \in R \) and \( \lambda_i, r \subseteq N (K_r), \forall r \in R \setminus R \)

COROLLARY 1. If \( G \) is a semisimple algebraic \( k \)-group split over \( K \), and \( r_g k r \) \( G > 0 \) for all \( r \in R \), then \( r_g k \) \( G > 0 \).

COROLLARY 2. If \( \tilde{G} \) is an admissible algebraic group over \( k \), and \( \{ \varphi_r : \tilde{G} \rightarrow G \}_{r \in R} \) is a consistent system of isomorphisms, then \( G \) and \( \tilde{G} \) are isomorphic over \( k \).
Proof of the Theorem. Let \( u_\alpha(t) \) and \( u_{\alpha',r}(t) \) be root subgroups with respect to the tori \( T \) and \( T_r \), respectively, with parameters \( t \) normalized as in [4] (pt. 3).

**Lemma.** There exists \( g \in G_{K_T} \) such that \( g u_\alpha(t) g^{-1} = u_{\alpha',r}(t) \).

There exists an \( m \in (\text{Aut } G)_{K_T} \) such that \( m(u_\alpha(t)) = u_{\alpha',r}(t) \). Since the systems \( \Delta_r \) are consistent, then, varying the numbering of the roots in \( \Delta \), we may suppose that \( m \) is an inner automorphism. Let \( M \) be a subgroup in \( (\text{ad } G)_{K_T} \) generated by the unipotent elements, and let \( D \) be the centralizer of the torus \( T_r \) in \( (\text{ad } G)_{K_T} \). Then \( [2] (\text{ad } G)_{K_T} = D \cdot M \), \( m = d \cdot m', d \in D, m' \in M \). The substitution of \( u_{\alpha',r}(t) \) for \( d^{-1}(u_{\alpha',r}(t)) \) corresponds to the substitution of the parameter \( t \). Having made this substitution we may assume that \( m \in M \). Since \( M \) is the unipotent part of \( G_{K_T} \), and since the substitutions which were made do not alter the conditions of the theorem, the lemma is proved.

Let \( B \) and \( B_r \) be Borel subgroups in \( G \) generated by the tori \( T \) and \( T_r \) and by the subgroups \( u_\alpha(t) \), \( \alpha \in \Delta \), and \( u_{\alpha',r}(t) \), \( \alpha \in \Delta_r \), respectively (\( r \in R \)). Let \( q \in R' \) be fixed, and let \( A_K(q) \) denote the adèlle product of the algebras \( K_r \) over all \( r \in R \setminus \{q\} \).

Now we take \( g = (g_r) \in G_{A_K(q)} \), \( g = 1 \) for all \( r \in R \setminus \{q\} \), \( g u_\alpha(t) g^{-1} = u_{\alpha',r}(t) \) for \( r \in \overline{R} \). Let \( T' = (T'_r) \) be an "\( A_K(q)\)-torus" in \( G \), where \( T_r' = T_r \) for \( r \in R \setminus \{q\} \), \( T'_r = T_r \) for \( r \in \overline{R} \). We define the "root subgroups" \( u_\alpha(t) \) and the "sets" \( \{\gamma_r\} \) analogously.

From the strong approximation theorem for \( G \) (over \( K \)), we may choose \( h \in G_K \) arbitrarily close to \( g \) in the topology of the group \( G_{A_K(q)} \). Let \( T' = hBh^{-1} \cap (hBh^{-1})_p \). Since \( T = B \cap B_r \), \( T_r = B_r \cap B_r' \), \( r \in \overline{R} \), the torus \( T' \) may be taken to be arbitrarily close to the torus \( T'' \). There exists an \( n \in U_K \) (the unipotent portion of the group \( B_k \)) such that \( T' = h(nTn^{-1})h^{-1} \). Since the tori \( hT'h^{-1} \) and \( T' \) are close, \( n \) is close to unity, and therefore, replacing \( h \) by \( hn \) if necessary, we obtain \( T' = hT'h^{-1} \).

Let \( u_\alpha'(t) = hu_\alpha(h^{-1}) \). We have

\[
\begin{align*}
  u_\alpha'(t) & = u_\alpha(h^{-1}t) = u_\alpha(t') \cdot u_\alpha(t'') \\
  u_{\alpha',r'}(t) & = u_{\alpha',r'}(t') = u_{\alpha',r'}(t'')
\end{align*}
\]

Since \( \sigma \) is a continuous operator, the closeness of \( u_\alpha'(t) \) and \( u_\alpha''(t) \) follows from the closeness of \( u_\alpha(t) \) and \( u_\alpha'(t) \). Choosing \( h \), we can say that \( \lambda_\alpha \) is arbitrarily close to \( \lambda_\alpha' \) and \( \lambda_\alpha'' \) is arbitrarily close to \( \lambda_\alpha'' \). This means that \( \lambda_\alpha \lambda_\alpha'^{-1} \) is arbitrarily close to \( 1 \). We have \( \lambda_\alpha \lambda_\alpha'^{-1} \subseteq U_r \subseteq N(K_r) \) for all \( r \in R \setminus \{q\} \), \( r \in R' \). For \( r \in \overline{R} \), \( \lambda_\alpha \lambda_\alpha'^{-1} \) lies in an arbitrarily small neighborhood of the identity of the field \( K_r \), and, in particular, \( \lambda_\alpha \lambda_\alpha'^{-1} \subseteq N(K_r) \) for all \( r \in \overline{R} \) as a consequence of the choice of \( R \) as a choice of \( \overline{R} \setminus R \).

Proof of Corollary 1. Let \( \overline{R} = R(T) \cup (R \setminus R) \cup R_\alpha \). Let \( \Delta \) denote the system of simple roots in \( G \), and let \( \delta \) denote a long root, \( \delta \in \Delta \). We will show that there exists an admissible \( k \)-torus \( T_r \) in \( G, r \in \overline{R} \), such that \( \lambda_\delta \subseteq N(K_r) \) for all \( r \in \overline{R} \). Actually, from sec. 9 of [4], \( \lambda_\delta \subseteq N(K_r) \) for some \( \delta \in \Delta \). If \( \delta \) is a long root, then by means of an element of the Weyl group we can transform \( \delta \) into \( \delta \). Hence, in this case we have \( \lambda_\delta \subseteq N(K_r) \). If \( \delta \) is a short root, then we can find a long root \( \gamma \) such that \( \Sigma' = (Q_\beta + Q_\gamma) \cap \Sigma \) is a system of roots of type \( G_2 \) or \( B_2 \). The corresponding group is isotropic; by using the classification of isotropic groups of this type it is easy to find in \( G(\Sigma') \) an admissible subtorus \( \overline{T} \), with respect to which \( \lambda_\delta \subseteq N(K_r) \) for all \( r \).

From the above we have \( \lambda_\delta \subseteq N(K_r) \) for all \( r \). Applying the theorem we establish our assertion.

Proof of Corollary 2. We take an admissible torus \( \overline{T} \) in \( G \) and let \( \overline{R} = R(T) \cup (R \setminus R) \cup R_\alpha \). Let \( \lambda_\alpha \) denote the set represented by the group \( \overline{G} \) with respect to the torus \( \overline{T} \). Let \( T_r = \varphi_r(T) \) for all \( r \in \overline{R} \), and apply the theorem. According to the theorem, \( G \) contains an admissible \( k \)-torus \( T' \) such that the set \( \{\lambda_r\} \) represented by the group \( G \) with respect to \( T' \) satisfies the conditions: \( \lambda_\alpha \subseteq \lambda_\alpha \cdot N(K_r) \) for all \( r \in \overline{R} \).

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**Literature Cited**