THE HASSE PRINCIPLE FOR ALGEBRAIC GROUPS SPLIT OVER A QUADRATIC EXTENSION

B. Yu. Veisfeiler

At the present time the Hasse principle is known to be valid for the majority of algebraic groups. However, its proof [1] makes use of the classification of algebraic groups. In this paper it will be shown that for groups which are split over a quadratic extension, the Hasse principle is a result of the strong approximation theorem for split groups and the Hasse principle for quaternions. We shall essentially use the approach given in [3, 4]. Let k be a field of algebraic numbers; K a quadratic extension, $\sigma \in \Gamma(K/k)$, $\sigma \neq 1$, R (respectively, R_a), the set of all valuations of k (respectively, Archimedian valuations of k). Furthermore, let U_r denote the identity of the field $k_r (r \in R)$, $R' = \{r \in R : K_r = k_r \oplus k_r\}$, $R'' = \{r \in R : K_r$ is a field$\}$, $R_n = \{r \in R'' : K_r/k_r$ is unramified$\}$. Let G be a semisimple simply connected algebraic group defined over k and split over K. As in [3, 4], we call a maximal subtorus in G "admissible" if it is defined and is anisotropic over k and split over K. Let $\{\lambda_\alpha\}_{\alpha\in\Sigma}$, $\lambda_\alpha \in \bar{k}^* \bmod N(K^*)$, denote the set being represented by the group G with respect to the admissible torus T. Let $R(T, \alpha) = \{r \in R: \lambda_\alpha \notin N(K_r)\}$, $R(T) = \bigcup R(T, \alpha)$, and let N denote the norm from K to k, and from K_r to k_r.

Definition. Let G, H be semisimple algebraic groups over k, and for each $r \in R$ let there be given a k_r-isomorphism φ_r: $G \to H$. We call the system $\{\varphi_r\}_{r\in R}$ "consistent" if for any class of parabolic subgroups $\mathscr{P}$ in G and $\mathscr{R}$ in H

$$\exists r \in R: \varphi_r(\mathscr{P}) = \mathscr{R} \Rightarrow \varphi_r(\mathscr{P}) = \mathscr{R} \quad \forall r \in R.$$

(If H = G and $\mathscr{R} = \mathscr{P}$, then the expression "system of consistent isomorphisms" is synonymous with the expression "system of inner automorphisms.")

Let T_r denote k_r-tori in G and let $\Delta_r = \{\alpha_{1,r}, \ldots, \alpha_{n,r}\}$ be an ordered system of simple roots in the root system of the group G with respect to T_r. We say that the system Δ_r is "consistent" if for all $i \in [1, n]$ there exists a class $\mathscr{P}_i$ of maximal parabolic subgroups in G such that in all Δ_r the root $\alpha_{i,r}$ corresponds to the class $\mathscr{P}_i$.

THEOREM. Let $\bar{R} \subset R''$, $\bar{R} \supset R(T) \cup (R'' \setminus R_n) \cup R_a$, $|\bar{R}| < \infty$. Let the group G be anisotropic over k, and let T_r, $r \in \bar{R}$, be an admissible k_r-torus in G. Let Δ_r be consistent systems of simple roots of G with respect to T_r, and let $\{\lambda_{\alpha,r}\}_{\alpha\in\Delta_r}$ be a set represented by the group G with respect to the torus T_r (over k_r). Then there exists an admissible k-torus T' and a system of simple roots Δ' with respect to T' satisfying the conditions:

a) $\Delta' = \{\alpha'_1, \ldots, \alpha'_n\}$ is consistent with Δ_r for all $r \in \bar{R}$;

b) if $\{\lambda'_\alpha\}_{\alpha\in\Delta'}$ is a set represented by the group G with respect to T', then $\forall i \in [1, n]$ we have $\lambda'_{\alpha_i} \in \lambda_{\alpha_i, r} \cdot N(K_r)$ $\forall r \in \bar{R}$ and $\lambda_{\alpha_i} \in N(K_r)$ $\forall r \in R \setminus \bar{R}$

COROLLARY 1. If G is a semisimple algebraic k-group split over K, and $\mathrm{rg}_{k_r} G > 0$ for all $r \in R$, then $\mathrm{rg}_k G > 0$.

COROLLARY 2. If $\tilde{G}$ is an admissible algebraic group over k, and $\{\varphi_r: \tilde{G} \to G\}_{r\in R}$ is a consistent system of isomorphisms, then G and $\tilde{G}$ are isomorphic over k.

Institute of Problems in Automation and Remote Control. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 6, No. 2, pp. 21-23, April-June, 1972. Original article submitted June 29, 1971.

Proof of the Theorem. Let $u_\alpha(t)$ and $u_{\alpha,r}(t)$ be root subgroups with respect to the tori T and T_r, respectively, with parameters t normalized as in [4] (pt. 3).

LEMMA. There exists $g_r \in G_{K_r}$ such that $g_r u_\alpha(t) g_r^{-1} = u_{\alpha,r}(t)$.

There exists an $m \in (\mathrm{Aut}\ G)_{K_r}$ such that $m(u_\alpha(t)) = u_{\alpha,r}(t)$. Since the systems Δ_r are consistent, then, varying the numbering of the roots in Δ, we may suppose that m is an inner automorphism. Let M be a subgroup in $(\mathrm{ad}\ G)_{K_r}$, generated by the unipotent elements, and let D be the centralizer of the torus T_r in $(\mathrm{ad}\ G)_{K_r}$. Then [2] $(\mathrm{ad}\ G)_{K_r} = D \cdot M$, $m = d \cdot m'$, $d \in D$, $m' \in M$. The substitution of $u_{\alpha,r}(t)$ for $d^{-1}(u_{\alpha,r}(t))$ corresponds to the substitution of the parameter t. Having made this substitution we may assume that $m \in M$. Since $M = \{\mathrm{ad}\ g, g \in G_{K_r}\}$ and since the substitutions which were made do not alter the conditions of the theorem, the lemma is proved.

Let B and B_r be Borel subgroups in G generated by the tori T and T_r and by the subgroups $u_\alpha(t)$, $\alpha \in \Delta$, and $u_{\alpha,r}(t)$, $\alpha \in \Delta_r$, respectively ($r \in \bar{R}$). Let $q \in R'$ be fixed, and let $A_K(q)$ denote the adèle product of the algebras K_r over all $r \in R \setminus \{q\}$.

Now we take $g = (g_r) \in G_{A_K(q)}$, $g = 1$ for all $r \in R \setminus \bar{R} \setminus \{q\}$, $g_r u_\alpha(t) g_r^{-1} = u_{\alpha,r}(t)$ for $r \in \bar{R}$. Let $T'' = (T''_r)$ be an "$A_K(q)$-torus" in G, where $T''_r = T$ for $r \in R \setminus \bar{R} \setminus \{q\}$, $T''_r = T_r$ for $r \in \bar{R}$. We define the "root subgroups" $u''_\alpha(t)$ and the "sets" $\{\lambda''_\alpha\}_{\alpha \in \Sigma}$ analogously.

From the strong approximation theorem for G (over K), we may choose $h \in G_K$ arbitrarily close to g in the topology of the group $G_{A_K(q)}$. Let $T' = hBh^{-1} \cap (hBh^{-1})^\sigma$. Since $T = B \cap B^\sigma$, $T_r = B_r \cap B_r^\sigma$, $r \in \bar{R}$, the torus T' may be taken to be arbitrarily close to the torus T''. There exists an $n \in U_K$ (the unipotent portion of the group B_K) such that $T' = h(nTn^{-1})h^{-1}$. Since the tori hTh^{-1} and T' are close, n is close to unity, and therefore, substituting, if necessary, h for hn, we obtain $T' = hTh^{-1}$.

Let $u'_\alpha(t) = hu_\alpha(t)h^{-1}$. We have

$$u'_\alpha(t)^\sigma = u'_{-\alpha}(\lambda'_\alpha t^\sigma), \quad u''_\alpha(t)^\sigma = u''_{-\alpha}(\lambda''_\alpha t^\sigma),$$
$$u'_{-\alpha}(t)^\sigma = u'_\alpha(\lambda'^{-1}_\alpha t^\sigma), \quad u''_{-\alpha}(t)^\sigma = u''_\alpha(\lambda''^{-1}_\alpha t^\sigma).$$

Since σ is a continuous operator, the closeness of $u'_\alpha(t)$ and $u''_\alpha(t)$ follows from the closeness of $u'_\alpha(t)^\sigma$ and $u''_\alpha(t)^\sigma$. Hence, choosing h, we can say that λ'_α is arbitrarily close to λ''_α and λ'^{-1}_α is arbitrarily close to λ''^{-1}_α. This means that $\lambda'_\alpha \lambda''^{-1}_\alpha$ is arbitrarily close to 1. We have $\lambda'_\alpha \lambda''^{-1}_{\alpha,r} \in U_r \subset N(K_r)$ for all $r \in R'' \setminus \bar{R}$ since $\bar{R} \supset R_n$, and $\lambda'_\alpha \lambda''^{-1}_{\alpha,r} \in k_r = N(K_r)$ for $r \in R'$. For $r \in \bar{R}$, $\lambda'_\alpha \lambda''^{-1}_{\alpha,r}$ lies in an arbitrarily small neighborhood of the identity of the field k_r, and, in particular, $\lambda'_\alpha \lambda''^{-1}_{\alpha,r} \in N(K_r)$, $r \in \bar{R}$. Our assertion follows from this since $\lambda''_{\alpha,r} \in N(K_r)$ for $r \in R \setminus \bar{R}$ as a consequence of the choice of $\bar{R}$ ($\bar{R} \supset R(T)$).

Proof of Corollary 1. Let $\bar{R} = R(T) \cup (R'' \setminus R_n) \cup R_\alpha$. Let Δ denote the system of simple roots in G, and let δ denote a long root, $\delta \in \Delta$. We will show that there exists an admissible k_r-torus T_r in G, $r \in \bar{R}$, such that $\lambda_{\delta,r} \in N(K_r)$ for $r \in \bar{R}$ (where $\{\lambda_{\alpha,r}\}$ is a set represented by the torus T_r). Actually, from sec. 9 of [4], $\lambda_{\beta,r} \in N(K_r)$ for some $\beta \in \Sigma$. If β is a long root, then by means of an element of the Weyl group we can transform β into δ. Hence, in this case we have $\lambda_{\delta,r} \in N(K_r)$. If β is a short root, then we can find a long root γ such that $\Sigma' = (Q\beta + Q\gamma) \cap \Sigma$ is a system of roots of type G_2 or B_2. The corresponding group is isotropic; by using the classification of isotropic groups of this type it is easy to find in $G(\Sigma')$ an admissible subtorus $\tilde{T}$, with respect to which $\lambda_{\beta,r} \in N(K_r)$ for the long root β.

From the above we have $\lambda''_{\delta,r} \in N(K_r)$ for all r. Applying the theorem we establish our assertion.

Proof of Corollary 2. We take an admissible torus $\tilde{T}$ in $\tilde{G}$ and let $\bar{R} = R(T) \cup R(\tilde{T}) \cup (R'' \setminus R_n) \cup R_a$. Let $\{\lambda_\alpha\}$ denote the set represented by the group $\tilde{G}$ with respect to the torus $\tilde{T}$. Let $T_r = \varphi_r(T)$ for all $r \in \bar{R}$, and apply the theorem. According to the theorem, G contains an admissible k-torus T' such that the set $\{\lambda'_\alpha\}$ represented by the group G with respect to T' satisfies the conditions: $\lambda'_\alpha \in \tilde{\lambda}_\alpha \cdot N(K_r)$ for $r \in R$ (since $\tilde{\lambda}_\alpha \in N(K_r)$) and $\lambda_\alpha \in N(K_r)$ for $r \in R \setminus \bar{R}$ (as a consequence of the choice of $\bar{R}$). Hence, by the theorem of global norms, $\lambda'_\alpha \in \tilde{\lambda}_\alpha \cdot N(K)$; i.e., G and $\tilde{G}$ represent the same set and therefore are k-isomorphisms.

The author wishes to thank E. B. Vinberg and D. A. Kazhdan for useful discussions.

LITERATURE CITED

1. G. Harder, "Über die Galois-Kohomologie halbeinfacher Matrizengruppen," Math. Z., 90, 404-428 (1965); 92, 396-415 (1965).

2. J. Tits, "Abstract and algebraic simple groups," Ann. Math., 80, 313-329 (1964).
3. B. Yu. Veisfeiler, "Notes on some algebraic groups," Funktsional' Analiz i Ego Prilozhen., 4, No. 1, 91 (1970).
4. B. Yu. Veisfeiler, "Semisimple algebraic groups which are split over a quadratic extension," Izv. Akad. Nauk SSSR, Ser. Mat., 35, 56-71 (1971).