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On the Structure of the Minimal Ideal of Some Graded Lie Algebras in Characteristic $p > 0$

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Inspired by Kostrikin and Shafarevich [5], a program of classification of finite dimensional simple Lie algebras in characteristic p involves two stages. First, one has to find a big and good subalgebra such that one would be able to associate to the pair (our algebra, this subalgebra) a long filtration. In particular, the subalgebra should contain an ideal consisting of nilpotent elements of the whole algebra. Then one has to prove that this subalgebra can be changed so that the corresponding filtration would remain long, but the associated graded Lie algebra would be of Cartan type.

The existence of parametrical families of simple Lie algebras constructed by Kac [4] (cf. also [8]) in all positive characteristics seems to imply that the passage to the graded Lie algebra is necessary in order to make the set of objects manageable.

In these notes we assume the existence of filtration and study the structural properties of the associated graded Lie algebra. The results are based on Block's theorem describing the structure of semisimple Lie algebras (cf. [1]).

We begin our study with a description of the construction of an uncondensable filtration in a Lie algebra (compare [6, 12]). The corresponding graded Lie algebra G may have a nilpotent graded ideal $M(G)$, the ideal of noneffectiveness. The quotient $G/M(G)$ is semisimple and has unique minimal ideal A, which is also graded. By the result of Block, $A = S \otimes B_n$, where S is a simple Lie algebra and $B_n = k[x_1, ..., x_n]/(x_1^p, ..., x_n^p)$, $[(s_1 \otimes b_1), (s_2 \otimes b_2)] = [s_1, s_2] \otimes b_1 b_2$. We show that the grading of A induces the gradings of S and B_n, such that $A_m = \sum_{i+j=m} S_i \otimes B_{n,j}$. Then the study breaks into two different cases, one of which ($A_1 = 0$) can be considered as degenerate and another ($A_1 \neq 0$) as non-degenerate. The filtration of B_n is trivial in the non-degenerate case and is non-trivial in the degenerate case. On the other hand the filtration of S is trivial in the degenerate case.

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To conclude this introduction we give two examples and state one conjecture. The examples show how (and that) both the nondegenerate case with $n \geqslant 1$ and the degenerate case occur in "real life." Of course, the examples we give are very special, but I do not have better ones at my disposal.

A DEGENERATE EXAMPLE. Let $G = \sum_{i \geqslant -2} G_i$ be a graded simple Lie p-algebra of Cartan type K_n. Set $L_0 = N_G(G_{-2})$. Then L_0 is a maximal subalgebra of G and G_{-2} is the nil-radical of L_0. Set $L = G$. Then (cf. Sect. 1.2 below) $L_1 = G_{-2}$, $L_2 = 0$ (since $L_1 = G_{-2}$ is the nilradical of L_0). Let us construct an uncondensable filtration in L (it is unique in our case). Let $\tilde{G} = \sum_{i \leqslant 1} \tilde{G}_i$ be the graded Lie algebra associated to the new filtration. We have $\tilde{G}_1 = L_1 = G_{-2}$. In particular $\dim \tilde{G}_1 = 1$. Further, $G_0 = H_{n-1} + T + D$, where H_{n-1} is the simple ideal which is of Cartan type H_{n-1}, D is an outer derivation of H_{n-1} of degree $(n-1)(p-1) - 2$ and T is a one-dimensional toral subalgebra (=the center of G_0). The subalgebra T acts nontrivially on $\tilde{G}_0$ and on $\tilde{G}_1$. On the other hand G_0 is not faithful on G_1 (because G_1 is one-dimensional and can faithfully afford only an action of one-dimensional algebra, namely that of T). Therefore (cf. Proposition 3.2.1(i) below) $[\tilde{G}_1, \tilde{G}_{-1}]$ does not contain T.

It is worth mentioning another pathology of our example: the ideal $M(\tilde{G})$ (cf. Sect. 1.5 below) is not zero. To describe $M(\tilde{G})$ we have to distinguish trivial subquotients for $[G_0, G_0]$ in G_i. They are of two different kinds. First, those which by commutation with G_{-2} can be brought to G_{-2}. They lie in G_{-2}, G_0, G_2,..., $G_{2(p-2)}$ and can be imagined as "forming" a Lie algebra W_1. Denote them by V_{-2}, V_0,... . Second, those which are translations by $(\operatorname{ad} G_{-2})^i$ of the $G_{(n+1)(p-1)-2}$ if $\dim K_n = p^n$ (i.e., $n + 3 \not\equiv 0(p)$) or of the exterior derivation of G of degree $(n+1)(p-1) - 2$ if $\dim K_n = p^n - 1$ (i.e., $n + 3 \equiv 0(p)$). Denote them by $V_{(n-1)(p-1)-2}$, $V_{(n-1)(p-1)}$,..., $V_{(n+1)(p-1)-4}$. They form a p-dimensional (respectively, $(p-1)$-dimensional) subspace which can be considered as a "module" for the "Lie algebra" of type W_1, constructed just before. In the first case we have $[G_{-2}, V_2] \subseteq G_0 \subset \tilde{G}_0$ with $V_2 \not\subset \tilde{G}_0$. Therefore V_2 and therefore V_{2i}, $0 < i < p-2$, are in $M(\tilde{G})$. Similarly, $V_{(n-1)(p-1)-2} \subset \tilde{G}_0$ and therefore $V_{(n-1)(p-1)}$,... are in $M(\tilde{G})$.

Now we have $\dim M(\tilde{G}) \geqslant 2p - 3 - \epsilon$ if $\dim K_n = p^n - \epsilon$, $\epsilon = 0$ or 1. On the other hand, (cf. Sect. 3.1 below) $\tilde{G}/M(\tilde{G}) = \tilde{G}_1 + T + D + H_{n-1} \otimes B_1$. So we have

$$\begin{aligned}\dim \tilde{G} &= 2p - 3 - \epsilon + 1 + 1 + 1 + p \cdot \dim H_{n-1} \\ &= 2p - \epsilon + p \cdot (p^{n-1} - 2) = p^n - \epsilon\end{aligned}$$

which is $\dim K_n$ as it should be. So $\dim M(\tilde{G}) = 2p - 3 - \epsilon$ and therefore $M(\tilde{G})$ is exhausted by above subspaces.

A NONDEGENERATE EXAMPLE (communicated to me by V. Kac; in fact Kac has found all gradings of Lie algebras of Cartan type for which G_{-1} is irreducible). Let G be a Lie p-algebra of Cartan type W_n. Identify W_n with Der B_n. Take $0 < s < n$ and ascribe weight -1 to $\partial_1, ..., \partial_s$ and weight zero to $x_{s+1}, ..., x_n$. This defines a grading of G and we have

$$G_{-2} = 0,$$
$$G_{-1} = \langle \partial_1, ..., \partial_s \rangle \otimes k[x_{s+1}, ..., x_n],$$
$$G_0 = \mathfrak{gl}(x_1, ..., x_s) \otimes k[x_{s+1}, ..., x_n] + \text{Der}\, k[x_{s+1}, ..., x_n],$$

etc. So we get a grading of depth 1. Of course, $G \supset (\text{Der}\, k[x_1, ..., x_s]) \otimes k[x_{s+1}, ..., x_n] = \tilde{G}$ with the grading induced by the standard grading of Der $k[x_1, ..., x_s] \simeq W_s$. So we have inside G the subalgebra $W_s \otimes B_{n-s}$. If $s > 1$ it is generated by G_0-invariant subspaces of G_1 and G_{-1}.

So although the whole algebra is simple, it contains a big and naturally defined subalgebra which is of nondegenerate type.

Our results and examples suggest the following conjecture:

Let $G = \sum_{i \geqslant -m} G_i$ be a gradded Lie algebra satisfying g1–g5 with $\mathscr{H} = 1$ (cf. Sect. 1.3) and such that $M(G) = 0$ (cf. Sect. 1.4). Suppose that G is generated by the subspace $G_1 + G_0 + G_{-1}$, that the representation of G_0 on G_1 is faithful and irreducible, and that the subalgebra generated by $G_{-1} + G_1$ is simple. Then G is either a classical Lie algebra, or a Lie p-algebra of Cartan type S_n or H_n with the standard or inverse to standard grading or a Lie algebra of Cartan type H_n with the "pathological" or inverse to it grading.

1. FILTERED LIE ALGEBRAS AND CORRESPONDING GRADED LIE ALGEBRAS

We shall consider a somewhat more general case than that considered in [2], [6]. We hope to be able to use this generality later.

1.1. *Properties of Filtered Lie Algebras*

A Lie algebra L with a descending sequence of subspaces $L = L_{-m} \supset L_{-m+1} \supset \cdots \supset L_0 \supset \cdots$ and with an automorphism group $\mathscr{H}$ (possibly $\mathscr{H} = 1$ even if $\text{Aut}\, L \neq 1$) is called $\mathscr{H}$-filtered if all L_i are $\mathscr{H}$-stable and the following conditions are satisfied:

f0. L_0 is a maximal $\mathscr{H}$-invariant subalgebra of L.

f1. $\bigcap L_i = 0$.

f2. $\dim L_i/L_{i+1} < \infty$.

f3. $[L_i, L_j] \subseteq L_{i+j}$.

f4. $\forall x \in L_i - L_{i+1}$, $i \geqslant 0$ $\exists y \in L_{-1}$ such that $[xy] \notin L_i$.

The filtration is called uncondensable if moreover

f5. $L_{i-1} = [L_{-1}, L_i] + L_i$, $\forall i < 0$.

f6. L_0/L_1 is represented faithfully on L_{-1}/L_0.

f7. L_{-1}/L_0 is irreducible for $(L_0/L_1, \mathscr{H})$ (that is, it has no proper subspaces which are both L_0/L_1- and $\mathscr{H}$-stable).

1.2. *Construction of a Filtration*

Let L now be a Lie algebra, let $\mathscr{H}$ be a subgroup of its automorphism group, and let L_0 be a maximal $\mathscr{H}$-invariant Lie subalgebra of L of finite codimension. We shall construct an uncondensable $\mathscr{H}$-filtration of L. Take in L/L_0 an arbitrary $(L_0, \mathscr{H})$-irreducible subspace, say S, and denote by L_{-1} its preimage in L. Set $L_{i-1} = [L_{-1}, L_i] + L_i$ for $i < 0$, $L_{i+1} = \{x \in L_i : [x, L_{-1}] \subset L_i\}$ for $i \geqslant 0$. Clearly the L_i's are $\mathscr{H}$-stable and the maximality of L_0 gives us $L = L_{-m}$ for some $m > 0$.

Proposition. *If L_0 does not contain any $\mathscr{H}$-stable ideals of L then the sequence of subspaces L_i, defined above, is an uncondensable filtration in L.*

Proof. f0, f4, f5, f7 are satisfied by definition.

Suppose that f3 is satisfied. Then $\bigcap L_i$ is an $\mathscr{H}$-stable ideal of L, contained in L_0, whence f1. Next, f2 is true by construction for $i < 0$. Suppose that it is true for all $i < d \geqslant 0$. Then $\dim L/L_d < \infty$. Since L_{d+1} is the kernel of the action of L_d on L_{-1}/L_d and since $\dim L_{-1}/L_d < \infty$, we get $\dim L_d/L_{d+1} \leqslant \dim \mathfrak{gl}(L_{-1}/L_d) = (\dim L_{-1}/L_d)^2 < \infty$. So f2 follows from f3. Also, since L_1 is the kernel of the action of L_0 on L_{-1}/L_0 we get f6.

It remains only to prove f3. It will be done in several steps

(i) f3 holds by construction for $i, j \leqslant 0$.

(ii) f3 holds by construction for $i = -1, j \geqslant 0$.

(iii) Let us prove by an induction that f3 holds for $i < 0, j \geqslant 0$.

Assume that $n < -1$ and that $[L_{n+1}, L_j] \subseteq L_{n+j+1}$ for all j. The beginning of the induction is $n = -2$ when our assumption holds by (i) and (ii). Let us prove that it holds for n. Since $L_n = [L_{-1}, L_{n+1}] + L_n$ and by (i), (ii) and the inductive assumption we have

$$\begin{aligned}
[L_n, L_{t+1}] &= [[L_{-1}, L_{n+1}] L_{t+1}] + [L_{n+1}, L_{t+1}] \\
&\subseteq [[L_{-1}, L_{t+1}] L_{n+1}] + [L_{-1}[L_{n+1}, L_{t+1}]] + L_{n+t+2} \\
&\subseteq [L_t, L_{n+1}] + [L_{-1}, L_{t+1+n+1}] + L_{n+t+2} \\
&\subseteq L_{n+t+1}.
\end{aligned}$$

This establishes our inductive step.

(iv) Let us prove f3 by an induction on $i + j$ for $i \geqslant 0, j \geqslant 0$. By (i) and (iii) f3 holds for $i + j \leqslant 0$. Assume that f3 holds for all i, j with $i + j < n \geqslant 1$. Take i, j with $i + j = n$. We have by the inductive assumption:

$$[L_i, L_j] \subseteq [L_i, L_{j-1}] \subseteq L_{i+j-1}$$

and by the inductive assumption together with (i) and (iii):

$$\begin{aligned}[L_{-1}[L_i, L_j]] &\subseteq [[L_{-1}, L_i] L_j] + [L_i[L_{-1}, L_j]] \\ &\subseteq [L_{i-1}, L_j] + [L_i, L_{j-1}] \subseteq L_{i+j-1}.\end{aligned}$$

Therefore by the definition of L_k for $k > 0$ we get that $[L_i, L_j] \subseteq L_{i+j}$ which completes our inductive step.

1.3. *The Associated Graded Lie Algebra*

Let $L = L_{-m} \supset L_{-m+1} \supset \cdots$ be an uncondensable $\mathscr{H}$-filtration in a Lie algebra L, where $\mathscr{H}$ is a subgroup of $\operatorname{Aut} L$. Put $G_i = L_i/L_{i+1}$, $G = \sum_{i \geqslant -m} G_i$. To define $[x, y]$, $x \in G_i$, $y \in G_j$, take $\tilde{x} \in L_i$, $\tilde{y} \in L_j$, whose images in G_i and G_j are x and y respectively and define $[x, y]$ to be the image of $[\tilde{x}, \tilde{y}] \in L_{i+j}$. So $[x, y] \in G_{i+j}$. Also for $h \in \mathscr{H}$ define hx to be the image of $h\tilde{x} \in L_i$ in G_i. It is clear then that we get a homomorphism $\mathscr{H} \to \operatorname{Aut} \sum G_i$ (where $\sum G_i$ is considered with operation [*, *] and with the natural structure of vector space). It is easy to check that the operation [*, *] makes G into a Lie algebra. It has the following properties, which are direct corollaries of properties f0–f7.

g1. $[G_i, G_j] \subseteq G_{i+j}$.

g2. $[G_i, G_{-1}] = G_{i-1}$ for $i < 0$.

g3. For any $x \in G_i$, $x \neq 0$, $i > 0$, $[x, G_{-1}] \neq 0$.

g4. G_{-1} is $(G_0, \mathscr{H})$-irreducible.

g5. G_{-1} is a faithful G_0-module.

We show below that a natural quotient of G by a graded ideal, contained in $\sum_{i<-1} G_i$ has also the property

g6. For any $x \in G_i$, $x \neq 0$, $i < 0$, $\exists j > 0$: $[x, G_j] \neq 0$.

In what follows we will use the following notation $G_+ = \sum_{i>0} G_i$, $G_- = \sum_{i<0} G_i$.

1.4. *Graded Algebras and Subtori in Their Automorphism Groups*

In this subsection we point out an evident (but useful) fact. Let M be an algebra. Suppose we are given a $\mathbf{Z}$-grading of M, i.e., $M = \oplus_{i \in \mathbf{Z}} M_i$, $M_i M_j \subseteq M_{i+j}$. Define then the action of the multiplicative algebraic group G_m on M in the following way. Identify the character group $X(G_m)$ of G_m with $\mathbf{Z}$ and let

$$gm = \chi_i(g)m \qquad \text{for} \quad m \in G_i$$

(i.e., for $\lambda \in \bar{k}^*$ we have $g_\lambda m = \lambda^i m$). Then our G_m becomes a subgroup of Aut M. On the other hand, suppose we are gievn a homomorphism $G_m \to \text{Aut } M$. Then $M = \bigoplus_{\chi \in X(G_m)} M_\chi$, the decomposition in the eigenspaces for G_m. Evidently we have (since $G_m \subseteq \text{Aut } M$) that $M_\chi M_\tau \subseteq M_{\chi+\tau}$. So if we identify $X(G_m)$ with $\mathbf{Z}$ we get a $\mathbf{Z}$-grading of M. Since there are only two distinct ways to identify $X(G_m)$ with $\mathbf{Z}$ we have the following

LEMMA. *The above correspondence is* $1 \div 1$ *correspondence between gradings of* M *and homomorphisms* $G_m \to \text{Aut } M$.

1.5. *The Radical of a Graded Lie Algebra*

Let $G = \sum_{i \geqslant -m} G_i$ be a graded Lie algebra satisfying g1, g3, g4. Put $M^0(G) = 0$

$$M^{i+1}(G) = \{x \in G_- \mid [G_+, x] \subset M^i(G)\}.$$

Set $M(G) = \bigcup M^i(G)$.

PROPOSITION. *$M(G)$ is an $\mathscr{H}$-invariant graded nil-ideal of G and $M(G/M(G)) = 0$. If the condition gi ($i = 1, 2, 3, 4$) holds in G it also holds in $G/M(G)$. If* g5 *holds in G and $G_{-1} \neq 0$ then it holds also in $G/M(G)$.*

Remark. The condition g6 holds in G if and only if $M(G) = 0$. In particular, it always holds in $G/M(G)$ if $G_+ \neq 0$.

Remark. In fact, $G/M(G)$ has no commutative $\mathscr{H}$-invariant ideals as will be shown in Corollary 1.6.4. In the finite dimensional case it does not have commutative ideals at all (cf. Corollary 1.6.5).

Proof. The fact that $M(G)$ is graded follows directly from 1.4 since $M(G)$ is invariant under all automorphisms which preserve the grading of G. In particular, it is invariant under $G_m \subset \text{Aut } G$ which defines the grading of G.

All other assertions are evident as soon as we establish that $M(G)$ is an ideal. Let us prove this.

Let U (resp., U_+, U_0, U_-) be the universal enveloping algebra of G (resp., of G_+, G_0, G_-). Then by the Poincaré–Birkhoff–Witt theorem, $U = U_- U_0 U_+$. We extend the adjoint representation of G to make U act on G.

We have from the definition that $U_+ \cdot M(G) \subseteq G_-$. Therefore, $U \cdot M(G) = U_- U_0 U_+ \cdot M(G) \subseteq U_- U_0 \cdot G_- \subseteq G_-$. Thus the ideal $U \cdot M(G)$ of G is contained in G_-.

Now let I be any ideal of G contained in G_-. Then G_+ acts nilpotently on I. Therefore (by Engel's theorem)

$$I = I_r \subseteq I_{r-1} \subseteq \cdots \subseteq I_0 \subseteq I_{-1} = 0$$

where $I_j = \{x \in I \mid [G_+, x] \subseteq I_{j-1}\}$ for $j \geqslant 0$. Then $I_j \supseteq M^j(G)$ for all $j \geqslant 0$ whence $M(G) \supseteq I$.

In particular, $U \cdot M(G) \subseteq M(G)$ so $M(G)$ is an ideal as asserted.

Remark. The proof shows that

$$M(G) = \{x \in G_- \mid U_+ x \subseteq G_-\} = \{x \in G_- \mid Ux \subseteq G_-\}$$
$$= \text{sum of all ideals of } G \text{ contained in } G_- .$$

1.6. *The Minimal Ideal of a Graded Algebra*

Let $G = \sum_{i \geqslant -m} G_i$ be a graded Lie algebra, satisfying g1–g5. Suppose, moreover, that $M(G) = 0$.

1.6.1. PROPOSITION. *Any $\mathscr{H}$-invariant non-zero ideal of G contains G_-.*

Proof. Let I be an ideal of G. Consider

$$I_r^{\#} = \left\{ x \in G_r \mid \exists x_i \in G_i ,\ i < r, \text{ with } x + \sum x_i \in I \right\}.$$

Then $I^{\#} = \sum I_r^{\#}$ is a graded ideal of G. Moreover, if I is $\mathscr{H}$-invariant then so is $I^{\#}$ and if $I^{\#} \supseteq G_-$ then so does I. Therefore it is sufficient to prove Proposition 1.6.1 for graded ideals. So we assume that I is graded and $\mathscr{H}$-invariant $I = \sum I_j$. If $I_j \neq 0$ for some $j \geqslant 0$ then by g3, g4, g5 we have $I_{-1} = G_{-1}$ whence by g2 we have $I \supseteq G_-$. If $I_j \neq 0$ for some $j < 0$ then by g6 (i.e., $M(G) = 0$) and g4 we have again $I_{-1} = G_{-1}$ whence, as above, $I \supseteq G_-$. This proves Proposition 1.6.1.

1.6.2. COROLLARY. *Any two nontrivial $\mathscr{H}$-invariant ideals of G intersect nontrivially.*

Proof. By Proposition 1.6.1 they intersect at least in $\sum_{i<0} G_i \neq 0$.

1.6.3. LEMMA. *Let N be an $\mathscr{H}$-invariant ideal of G_0, $N \neq 0$.*

(i) *N does not act nilpotently on G_{-1}.*

(ii) *The algebra $N + \sum_{i<0} G_i$ is not nilpotent.*

Proof. Suppose that N acts nilpotently. Set $V = \{x \in G_{-1} \mid [N, x] = 0\}$. Our assumption implies that $V \neq 0$. Since $\mathscr{H} \subseteq \operatorname{Aut} G$ and $\mathscr{H} N \subseteq N$, V is an $\mathscr{H}$-invariant subspace of G_{-1}. We have

$$[[G_0 , V]N] \subseteq [[G_0 , N]V] + [G_0[N, V]] \subseteq [N, V] + 0 = 0.$$

Therefore, $[G_0 , V] \subseteq V$. So V is an $(G_0 , \mathscr{H})$-invariant subspace of G_{-1}. Since $V \neq 0$ we have from g4 that $V = G_{-1}$, a contradiction with g5. This proves (i).

To prove (ii) suppose that $M = N + \sum_{i<0} G_i$ is nilpotent. Then $\sum_{i<0} G_i$ is an ideal of M and therefore N acts on M. Since M is nilpotent this action is nilpotent, a contradiction with (i).

1.6.4. COROLLARY. *G does not contain any nontrivial $\mathscr{H}$-invariant commutative ideals.*

Proof. Let R be an ideal of G. Then $R \supset G_{-1}$ by 1.6.1. Therefore $R \supset [G_{-1}, G_1]$. Clearly $[G_{-1}, G_1]$ is an $\mathscr{H}$-invariant ideal of G_0. By 1.6.3 the Lie algebra $M = \sum_{i<0} G_i + [G_{-1}, G_1]$ is not nilpotent. But $M \subseteq R$. Therefore R is not nilpotent. In particular, it is not commutative.

1.6.5. COROLLARY. *If G is finite-dimensional it is semi-simple.*

Proof. Let R be the radical of G. Clearly $\mathscr{H}R \subseteq R$. If $R \neq 0$, then R contains a nonzero $\mathscr{H}$-stable abelian ideal of G (e.g., the last nonzero term of the derived series of R). This contradicts Corollary 1.6.4. Therefore $R = 0$ as asserted.

1.6.6. COROLLARY. *If $\mathscr{H}$ does not have nontrivial finite quotient groups and if* $\dim G < \infty$, *then G contains a unique minimal ideal. This ideal is $\mathscr{H}$-stable.*

Proof. Since $\dim G < \infty$, then by 1.6.5 the situation is trivial if char $k = 0$. So we have to assume that char $k \neq 0$. Then by Lemma 9.1 of [1] (with $D = 0$) we know that G has finitely many minimal ideals. Therefore $\mathscr{H}$ has to permute them. But $\mathscr{H}$ does not have finite quotients. Therefore $\mathscr{H}$ fixes every one of them. Therefore they intersect nontrivially by 1.6.2. Since Lemma 9.1 of [1] says that the sum of minimal ideals of G is direct, we must have that their number is one. It follows from the preceding argument that this ideal is $\mathscr{H}$-stable.

1.6.7. THEOREM. *Suppose that k is algebraically closed. Let A be the intersection of all nonzero $\mathscr{H}$-stable ideals of G.*

(i) *$A \neq 0$; A is the unique nonzero minimal $\mathscr{H}$-stable ideal of G.*

(ii) *A is a graded ideal of G; it is stable under $N_{\mathrm{Aut}\, G}(\mathscr{H})$.*

(iii) *If* $\dim G < \infty$ *and if $\mathscr{H}$ does not have nontrivial finite quotients, then $A \simeq S \otimes B_n$, where S is a simple Lie algebra and B_n is the quotient of the ring of polynomials in $x_1, \ldots, x_n$ by the ideal generated by $x_1^p, \ldots, x_n^p$ and $p =$* char k *if* char $k \neq 0$, $p = 1$ *if* char $k = 0$.

Proof. Since A contains $\sum_{i<0} G_i$ (by Proposition 1.6.1) we have $A \neq 0$. Then (i) is clear.

To prove (ii) notice first that if $g \in \mathrm{Aut}\, G$ and H is an $\mathscr{H}$-stable ideal of G, then $g(H)$ is an $g\mathscr{H}g^{-1}$-stable ideal of G. Therefore, A is $N_{\mathrm{Aut}\, G}(\mathscr{H})$-stable. Let $\varphi \colon G_m \to \mathrm{Aut}\, G$ determine (cf. 1.4) the grading of G. Then $\varphi(G_m)$ commutes with $\mathscr{H}$, therefore $\varphi(G_m) \subset N_{\mathrm{Aut}\, G}(\mathscr{H})$. Therefore A is $\varphi(G_m)$-stable, i.e., it is graded. So (ii) is proved.

Let us prove (iii). If char $k = 0$, then by 1.6.5 and 1.6.6, A is simple, i.e., $A = S \otimes B_n$, $B_n = k$ in our case. If char $k = p > 1$, then by Theorem 9.3 of [1] and by Corollary 1.6.6 we again get our assertion.

2. Preliminary Remarks on the Structure of the Minimal Ideal of G

Let G be a graded Lie algebra, satisfying g1–g6. Assume from now on that dim $G < \infty$, k is algebraically closed, char $k = p > 1$, $\mathscr{H} = \{1\}$. Let $A = A(G)$ be the minimal ideal of G (cf. Corollaries 1.6.5, 1.6.6). We know by Corollary 1.6.6 that

$$A = S \otimes B_n .$$

We set $C = B_n$, $C_1 =$ the maximal ideal of B_n (i.e., $C_1 = \sum_{i=1}^{n} x_i B_n$), $C_i = C_1{}^i$). We set next $N_i = S \otimes C_i \subseteq A$, $i = 1, 2, \ldots$, $N = N_1$.

2.1. Lemma. (i) *$N = S \otimes C_1$ is the radical of $A = S \otimes C$.*

(ii) *N is the nilradical of A.*

(iii) *$A/N = S$.*

(iv) *N is a graded ideal of A.*

(v) *$N_{i+1} = [N, N_i]$, $i = 1, 2, \ldots$.*

Proof. Clearly, $S \otimes C_1$ is a nil-ideal of A. Since $A/N = (S \otimes C)(S \otimes C_1) = S \otimes k \simeq S$ is simple we have at once (i), (ii), (iii). By (i), N is invariant under all automorphisms of A. Therefore (iv) follows from 1.4. The last assertion (v) follows from the simplicity of S and from the fact that $C_{i+1} = C_1 \cdot C_i$.

2.2. *On the Automorphism Group of $S \otimes B_n$.*

The group $\mathscr{H} = \mathrm{Aut}\, S \times \mathrm{Aut}\, B_n$ acts on $S \otimes B_n$ by the formula

$$(g, h)(s \otimes b) = (g(s) \otimes h(b)).$$

Clearly, this defines $\mathscr{H}$ as a subgroup of $\mathrm{Aut}(S \otimes B_n)$. Let us denote by ψ the corresponding imbedding of $\mathscr{H}$ into $\mathrm{Aut}(S \otimes B_n)$.

On the other hand, any $h \in \mathrm{Aut}(S \otimes B_n)$ induces an automorphism $\varphi_1(h)$ of $S = (S \otimes B_n)/N$ (since N is an ideal stable under all automorphisms) and an automorphism $\varphi_2(h)$ of B_n (since $B_n = Z_{\mathrm{End}(S \otimes B_n)}(\mathrm{ad}(S \otimes B_n))$). Denote by $\mathscr{N}$ the group $\mathrm{Ker}\, \varphi_1 \cap \mathrm{Ker}\, \varphi_2$. Since ψ is a section of $\varphi_1 \times \varphi_2$, we have

$$\mathrm{Aut}(S \otimes B_n) = \mathscr{N} \rtimes \psi(\mathscr{H})$$

($\mathscr{N}$ is a normal subgroup, $\rtimes$ stands for a semidirect product).

2.2.1. LEMMA. *$\mathcal{N}$ is a unipotent group.*

Proof. Let $L_b \in \operatorname{End}(S \otimes B_n)$ be the operator of multiplication by $b \in B_n$. Then for $h \in \operatorname{Aut}(S \otimes B_n)$ we have (by definition) $L_{\varphi_2(h)b} = hL_b h^{-1}$. If $s \in S$ and $b, b' \in B_n$ we have therefore,

$$h(s \otimes bb') = hL_b(s \otimes b') = hL_b h^{-1}h(s \otimes b') = L_{\varphi_2(h)b}h(s \otimes b')$$

Now take $h \in \mathcal{N} = \operatorname{Ker} \varphi_1 \cap \operatorname{Ker} \varphi_2$. Since $h \in \operatorname{Ker} \varphi_1$, we have $h(s \otimes 1) = s \otimes 1 \bmod N$ (where $N = S \otimes C_1$). Let $\{x^e = x_1^{e_1} \cdots x_n^{e_n}\}$ with $0 \leqslant e_i < p$ be a basis of B_n. Set $| e | = \sum e_i$.

Let us prove by induction that

$$h(s \otimes x^e) = s \otimes x^e \bmod(S \otimes C_{|e|+1}).$$

As remarked above it is true if $| e | = 0$. Suppose we have proved this for all admissible e with $| e | < m$. Take x^e with $| e | = m$ and write $x^e = x_i \cdot x^f$ where $| f | = | e | - 1$. Then

$$\begin{aligned} h(s \otimes x^e) &= h(s \otimes x_i x^f) = hL_{x_i}(s \otimes x^f) \\ &= L_{\varphi_2(h)x_i}h(s \otimes x^f) \in L_{\varphi_2(h)x_i}(s \otimes x^f + S \otimes C_{|f|+1}) \end{aligned}$$

where the last inclusion uses our induction assumption. Since $\varphi_2(h) = 1$ the last term turns into $s \otimes x^e + S \otimes C_{|e|+1}$. The fact we have just established says that $h \in \mathcal{N}$ acts as a unipotent matrix on $S \otimes B_n$, which proves our Lemma.

2.2.2. In fact the group $\operatorname{Ker} \varphi_1 \cap \operatorname{Ker} \varphi_2$ can be constructed explicitly.

LEMMA. *For $x \in \operatorname{Der} S$ and $c \in C_1$ the exponential $\exp(x \otimes c)$ is defined and belongs to $\mathcal{N}$.*

Proof. It is defined since

$$\exp(x \otimes c) = \sum_{i=0}^{p-1} \frac{x^i \otimes c^i}{i!}$$

because $c^p = 0$. Clearly, $\exp(x \otimes c)$ acts trivially both on $S = (S \otimes B_n)/N$ and on $B_n = Z_{\operatorname{End}(S \otimes B_n)}(\operatorname{ad}(S \otimes B_n))$. Therefore the only thing to prove is that $\exp(x \otimes c) \in \operatorname{Aut}(S \otimes B_n)$.

For any $s_1, s_2 \in S$, $c_1, c_2 \in C$ we have

$$\begin{aligned} &[(x \otimes c)^i(s_1 \otimes c_1), (x \otimes c)^j(s_2 \otimes c_2)] \\ &\quad = [x^i(s_1) \otimes c^i c_1, x^j(s_2) \otimes c^j c_2] \\ &\quad = [x^i(s_1), x^j(s_2)] \otimes c^{i+j} c_1 c_2. \end{aligned}$$

This expression is zero whenever $i + j \geqslant p$ (since $c^p = 0\ \forall c \in C_1$). Therefore our assertion follows from [7, Lemma 1.2].

2.2.3. COROLLARY. *Any subtorus of* $\mathrm{Aut}(S \otimes B_n)$ *is conjugate in* $\mathrm{Aut}(S \otimes B_n)$ *to a subtorus of* $\psi(\mathscr{H})$.

Proof. Let $\mathscr{T}$ be a maximal torus of $\mathscr{H}$. Then by 2.2.1 $\psi(\mathscr{T})$ is a maximal torus of $\mathrm{Aut}(S \otimes B_n)$. Now our assertion follows from the conjugacy of maximal tori in algebraic groups.

2.2.4. PROPOSITION. *There exist a Lie algebra grading* $S = \bigoplus_{i=q}^{r} S_i$, *and an algebra grading* $B_n = \sum_{i=t}^{m} B_{n,i}$ *such that*

$$A_s = \sum_{i+j=s} S_j \otimes B_{n,i}\,.$$

N.B. The symbols r, q, t, m are fixed from this point until the end of Section 4.

Proof. Let $\mathscr{T}$ be the one-dimensional torus which defines (by 1.4) the grading of A. By Corollary 2.2.3 we can assume that $\mathscr{T}$ preserves S. Then the action of $\mathscr{T}$ determines a grading of S. On the other hand, $\mathscr{T}$ acts on $B_n = Z_{\mathrm{End}(S\otimes B_n)}(\mathrm{ad}(S \otimes B_n))$. Therefore it induces the grading of B_n. Since $S \otimes B_n$ is a free B_n-module generated by $S \otimes 1$, we get our assertion.

2.3. *Some Properties of the Gradings* $S = \sum S_i$, $B_n = \sum B_{n,i}$

2.3.1. LEMMA. *The grading of S determined by Proposition* 2.2.4 *coincides with the grading of the quotient Lie algebra* $S = (S \otimes B_n)/N$ *of the graded Lie algebra* $S \otimes B_n$ *by the graded ideal* N.

Evident.

2.3.2. LEMMA. $\sum_{i\neq 0} B_{n,i} \subseteq C_1$.

Proof. Every element of $B_{n,i}$, $i \neq 0$, is nilpotent and every nilpotent element of B_n belongs to C_1.

2.3.3. LEMMA. (i) $q \leqslant 0$, $r \geqslant 0$. (ii) $q < 0$ *if and only if* $r > 0$.

Proof. If $q \geqslant 0$, then $S_+ = \sum_{i>0} S_i$ is a nilpotent ideal of S. Therefore $r = 0$ in this case. Since the same argument holds for r, we get our assertion.

3. DEGENERATE CASE

3.1. THEOREM. *Let* $G = \sum G_i$ *be a graded Lie algebra, satisfying* g1–g6 *with* $\mathscr{H} = 1$. *Let* $A = A(G)$ *be the minimal ideal of* G. *Suppose that* $A_1 = 0$. *Then*

(i) *There exist a simple Lie algebra S and integers u, v such that*

$$A = S \otimes B_{u+v},$$

(ii) $A_0 = [G_1, G_{-1}] = S \otimes B_u$.

(iii) $A_i = G_i = S \otimes B_u \otimes B_{v,i}$ $\forall i < 0$, *where $B_{v,i}$ is the subspace of homogeneous polynomials of degree $-i$ in $B_v = k[x_1, ..., x_v]/(x_1^p, ..., x_v^p)$.*

(iv) *If W_v is the Lie algebra of derivations of B_v and if $W_v = \sum_{i \leqslant 1} W_{v,i}$ is the grading induced by the given grading of B_v, then*

$$G_0 \subseteq \mathrm{Der}(S \otimes B_u) + 1_S \otimes B_u \otimes W_{v,0}, \qquad W_{v,0} \simeq \mathfrak{gl}(B_{v,-1}).$$

Moreover, $G_i \simeq S \otimes B_u \otimes B_{v,i}$ as G_0-modules for $i < 0$.

(v) $G_1 \subseteq B_u \otimes W_{v,1}$, *$G_1$ is a trivial $[G_1, G_{-1}]$-module.*

(vi) $G_2 = 0$. *In particular*, $[G_1, G_1] = 0$.

Proof. We have by Corollary 1.6.6 and Proposition 2.2.4 that

$$A = S \otimes B_n, \qquad A_s = \sum_{i+j=s} S_i \otimes B_{n,j}.$$

Let r, m, q, t be as in Proposition 2.2.4. Since $A_1 = 0$ it follows from g3 that $A_i = 0$ for $i > 0$. Therefore $r = m = 0$. By Lemma 2.3.3 we have also $q = 0$. Therefore

$$A_i = S \otimes B_{n,i}.$$

By g2 we have $B_{n,i} = B_{n,-1}^{-i}$ for $i < 0$. Therefore the torus $\mathscr{T}$ defining the grading of B_n can be assumed to preserve the space $\sum_{i=1}^{n} kx_i$; then $\mathscr{T}$ has on this space at most two characters, namely 0 and -1. One can choose the x_i in such a way that $\mathscr{T}$ fixes $x_{v+1}, ..., x_n$ and has the character -1 on $\sum_{i=1}^{v} kx_i$. Set $u = n - v$. Then $B_n = B_u \otimes B_v$, $B_{n,i} = B_u \otimes B_{v,i}$. This establishes (i), (ii), (iii).

Since A is an ideal in G we have the natural mapping $G \to \mathrm{Der}\, A$. We have (by [1])

$$\mathrm{Der}(S \otimes B_n) = (\mathrm{Der}\, S) \otimes B_n + 1_S \otimes \mathrm{Der}\, B_n.$$

The torus $\mathscr{T}$, corresponding to the grading, acts on Der A. It acts trivially on $S \otimes B_u$. Therefore

$$(\mathrm{Der}(S \otimes B_n))_0 = \mathrm{Der}(S \otimes B_u) + 1_S \otimes B_u \otimes (\mathrm{Der}\, B_v)_0,$$
$$(\mathrm{Der}(S \otimes B_n))_1 = 1_S \otimes B_u \otimes (\mathrm{Der}\, B_v)_1.$$

Since $(\mathrm{Der}\, B_v)_0 = \mathfrak{gl}(B_{v,-1})$, the first of these equalities gives us (iv). The second one gives the first part of (v). The second part of (v) follows from (ii) since S commutes with B_n.

Now (vi) follows from g3 and the fact that $A_1 = 0$ ($[A_{-1}, G_2] \subseteq A_1 = 0$).

3.2. *An Application of Theorem* 3.1

3.2.1. PROPOSITION. *Let G be a graded Lie algebra, satisfying* g1–g5. *Let V be a G_0-submodule of G_1. Suppose that G_0 is not faithful on V. Then*

(i) $[V, V] = 0$, $[[G_{-1}, V]V] = 0$.

(ii) *Let $\bar{G}$ be the subalgebra of G generated by $G_{-1} + G_0 + V$. Then $\bar{G}/M(\bar{G})$ satisfies the assumptions and conclusions of Theorem* 3.1.

3.2.2. LEMMA. *Let $H = \bigoplus_i H_i$ be a graded Lie algebra, satisfying* g1, g4, g5. *Let $\tilde{H}_0$ be the kernel of the action of H_0 on H_1. If $\tilde{H}_0 \neq 0$, then $[H_1, H_{-1}] \subseteq \tilde{H}_0$.*

Proof. Since $\tilde{H}_0$ is an ideal of H_0 we have $[[H_1, H_{-1}]\tilde{H}_0] \subseteq \tilde{H}_0$. On the other hand $[[H_1, H_{-1}]\tilde{H}_0] = [H_1[H_{-1}, \tilde{H}_0]]$.

Let us show that $[H_{-1}, \tilde{H}_0] = H_{-1}$. Let $[H_{-1}, \tilde{H}_0] = V \subseteq H_{-1}$. Then

$$[H_0, V] = [H_0[H_{-1}, \tilde{H}_0]] \subseteq [[H_0 H_{-1}], \tilde{H}_0] + [H_{-1}, [H_0, \tilde{H}_0]] \subseteq V.$$

We have therefore by g4: $V = 0$ or G_{-1}. By g5 the case $V = 0$ is impossible.

Therefore $[H_{-1}, \tilde{H}_0] = H_{-1}$, i.e., $\tilde{H}_0 \supseteq [[H_1, H_{-1}]\tilde{H}_0] = [H_1, H_{-1}]$, as asserted.

3.2.3. LEMMA. *Let $H = \bigoplus H_i$ be a graded Lie algebra, satisfying* g1, g3, g4, g5. *Let $\tilde{H}_0$ be the kernel of H_0 on H_1. If $\tilde{H}_0 \neq 0$, then $[H_1, H_1] = 0$.*

Proof. $[H_{-1}[H_1, H_1]] \subseteq [H_1[H_{-1}, H_1]] = 0$ by 3.2.2. By g3, $[H_1, H_1] = 0$.

3.2.4. *Proof of Proposition* 3.2.1. We can assume that $V = G_1$, $G = \bar{G}/M(\bar{G})$. Then 3.2.2 and 3.2.3 imply (i). But it follows from (i) that $A(G) \subseteq \sum_{i \leqslant 0} G_i$, i.e., $A_1 = 0$, i.e., Theorem 3.1 is applicable. It gives us (ii).

4. NONDEGENERATE CASE

4.1. THEOREM. *Let G be a graded Lie algebra satisfying* g1–g6. *Let $A = A(G)$ be the minimal ideal of G. Suppose that $A_1 \neq 0$.*

(i) *There exist a simple graded Lie algebra $S = \otimes S_i$ and an integer n such that*

$$A = S \otimes B_n, \qquad A_i = S_i \otimes B_n.$$

In particular, $G_i = S_i \otimes B_n$ for $i < 0$.

(ii) *If* Der $S = \oplus$ (Der $S)_i$ *is the grading of the derivation algebra of* S *induced by the grading of* S, *then*

$$G_0 \subseteq (\text{Der } S)_0 \otimes B_n + 1_S \otimes \text{Der } B_n ,$$
$$G_i \subseteq (\text{Der } S)_i \otimes B_n \qquad \text{for} \quad i > 0.$$

(iii) *The graded Lie algebra* $S = \sum S_i$ *staisfies* g1, g2, g3, g5. *The graded Lie algebra* $S = \sum S_i + (\text{Der } S)_0$ *satisfies also* g4.

4.2. In the proof of this theorem we assume that B_n and S are given gradings as in Proposition 2.2.4:

$$S = \bigoplus_{q \leqslant i \leqslant r} S_i , \qquad B_n = \bigoplus_{t \leqslant i \leqslant m} B_{n,i} , \qquad A_i = \sum_{\alpha+\beta=i} S_\alpha \otimes B_{n,\beta} ,$$

and that conditions of Theorem 4.1 hold. We also use notations C_i, N_i, C, N of the beginning of Section 2.

For convenient reference we restate below some of the results of the preceding sections.

LEMMA. (i) $S_j \otimes B_{n,i}$, $i \neq 0$, *is contained in the nilradical* N *of* A.

(ii) *Neither* A_1, *nor* $A_{-1} = G_{-1}$ *is contained in* N.

Proof. (i) follows from 2.1 and 2.3.2. To prove (ii) observe that $[G_0, A_1] \subseteq A_1$, $[G_0, A_{-1}] \subseteq A_{-1}$. Therefore $[A_1, A_{-1}]$ is an ideal in G_0. If A_1 or A_{-1} is contained in N, then $[A_1, A_{-1}] \subset N$ (since N is an ideal of A). We have $[A_1, A_{-1}] \neq 0$ by g3. Now by g4 we have $[[A_1, A_{-1}]\, A_{-1}] = A_{-1}$. Then by g2 the assumption $A_1 \subset N$ or $A_{-1} \subset N$ implies that $[A_1, A_{-1}] + \sum_{i<0} A_i$ is contained in N, and therefore $[A_1, A_{-1}]$ acts nilpotently on A_{-1}, a contradiction with the equality $[[A_1, A_{-1}]\, A_{-1}] = A_{-1}$.

Remark. One can also use Lemma 1.6.3.

4.3. LEMMA. (i) $q < 0, r > 0$.

(ii) $m = 0$.

(iii) $S_1 \neq 0$, $S_{-1} \neq 0$.

(iv) $[S_i, S_{-1}] = S_{i-1}$ *for* $i \leqslant 0$.

Proof. Assertion (i) follows from (iii) proved below.

To prove (ii) suppose that $m > 0$. Then by 4.2(i), $N \supset A_{r+m} = S_r \otimes B_{n,m} \neq 0$. Since $A_{-1} = G_{-1}$ we have by g3, g5 that $0 \neq (\text{ad}\, A_{-1})^{r+m+1} A_{r+m} \subset N$. Since A_{-1}, A_{r+m} are G_0-invariant we have from g4 and from the above inclusion that $G_{-1} = A_{-1} \subset N$, a contradiction with 4.2(ii).

Since $A_{-1} \not\subset N$, $A_1 \not\subset N$ by 4.2(ii), it follows from 4.2(i) that $S_{-1} \neq 0$, $S_1 \neq 0$. This proves (iii).

The last assertion (iv) follows immediately from Lemma 2.3.1 and 4.2(i).

4.4. PROPOSITION. $B_n = B_{n,0}$ (*i.e.*, $m = t = 0$).

Proof. We know (by Lemma 4.3(ii)) that $m = 0$. So it remains to prove that $t = 0$. Suppose $t \neq 0$. Let $s > 0$ be minimal such that $B_{n,-s} \neq 0$. Then

$$A_{q-s} = \sum_{a+b=q-s} S_a \otimes B_{n,b}$$

$$= \sum_{\substack{a+b=q-s \\ a \geqslant q, b \leqslant -s}} S_a \otimes B_{n,b} \, .$$

Therefore by 4.2(i) we have $A_{q-s} \subseteq N$.

Let I be the ideal of A generated by A_{q-s}. Since A_{q-s} and A are G_0-invariant so is I. Also $I \subseteq N$ since $A_{q-s} \subseteq N$. Remark next that $A = S \otimes B_n \supseteq S \otimes k$, that the ideal of S generated by S_q is S itself and that $A_{q-s} \supseteq S_q \otimes B_{n,-s}$. It follows from the above remark that $I \supset S \otimes B_{n,-s} \supseteq S_r \otimes B_{n,-s} \neq 0$. In particular, $I_{r-s} \neq 0$.

We claim now that $r - s \geqslant -1$. Indeed if $r - s < -1$ then $G_{-1} = S_{-1} \otimes B_{n,0}$ and then $G_i = S_i \otimes B_{n,0}$ for all $i < 0$. But then $t = 0$, a contradiction.

So $r - s \geqslant -1$. Since $I_{r-s} \neq 0$ we have by g3 and g5 that $I_{-1} \neq 0$. Since I is G_0-invariant, we have $I_{-1} = A_{-1} = G_{-1}$. Thus $N \supseteq G_{-1}$, a contradiction with 4.2(ii).

4.5. *Proof of Theorem* 4.1

The assertion (i) follows from Proposition 4.4 and 2.2.4. Since A does not have center we have an imbedding

$$\varphi \colon A \hookrightarrow \operatorname{Der} A.$$

Since A is an ideal of G we have an imbedding

$$\psi \colon G \hookrightarrow \operatorname{Der} A.$$

The action of the torus $\mathscr{T} \subseteq \operatorname{Aut} G$, which defines the gradings of A and G (by 1.4), defines also the grading of Der A. The action of $\mathscr{T}$ commutes with φ and ψ. Therefore

$$\psi(G_i) \subseteq (\operatorname{Der} A)_i \, .$$

We have $\operatorname{Der} A = \operatorname{Der} S \otimes B_n + 1_S \otimes \operatorname{Der} B_n$. Since $\mathscr{T}$ acts trivially on B_n we have

$$1_S \otimes \operatorname{Der} B_n \subseteq (\operatorname{Der} A)_0 .$$

Therefore

$$(\operatorname{Der} A)_i = (\operatorname{Der} S)_i \otimes B_n \qquad \text{for} \quad i \neq 0,$$
$$(\operatorname{Der} A)_0 = (\operatorname{Der} S)_0 \otimes B_n + 1_S \otimes \operatorname{Der} B_n .$$

This proves (ii).

To prove (iii) we will use the following general lemma.

4.5.1. LEMMA. *Let $H = \oplus_{-v\leqslant i\leqslant u} H_i$ be a graded Lie algebra satisfying* g1, g2. *If H is simple then*

(i) g3, g5 *hold in H and $H_0 \neq 0$.*

(ii) *H_{-v} is irreducible for H_0.*

(iii) $H_{-v+i} = \sum_{\alpha_j>0, \sum_{1\leqslant j\leqslant s}\alpha_j=i} (\operatorname{ad} H_{\alpha_1}) \cdots (\operatorname{ad} H_{\alpha_s}) H_{-v}$.

Proof. Set $N^0(H) = \tilde{N}^0(H) = 0$,

$$N^{i+1}(H) = \left\{h \in \sum_{i\geqslant 0} H_i \mid [H_{-1}, h] \subseteq N^i(H)\right\},$$

$$\tilde{N}^{i+1}(H) = \left\{h \in \sum_{i\geqslant 0} H_i \mid \left[\sum_{i<0} H_i , h\right] \subseteq \tilde{N}^i(H)\right\},$$

$$N(H) = \bigcup N^i(H), \qquad \tilde{N}(H) = \bigcup \tilde{N}^i(H).$$

It follows from g2 that $N(H) = \tilde{N}(H)$. Then the same argument as in the proof of 1.5 shows that $N(H)$ is an ideal of H. Therefore $N(H) = 0$, i.e., $g3$, g5 hold in H. If $H_0 = 0$ then every element of H is nilpotent which is impossible by Engel's theorem. This proves (i). Let $V \neq 0$ be a H_0-invariant subspace in H_{-v}. Set recursively $H'_{-v} = V$,

$$H'_{-v+i} = \sum_{0<j\leqslant i} (\operatorname{ad} H_j)\, H'_{-v+i-j} .$$

We have by induction on i, beginning with $i = -v$, that

$$[H_0 , H'_i] \subseteq H'_i .$$

We have $[H_{-1}, H'_{-v}] = 0$. Let us prove by induction that $[H_{-1}, H'_{-v+i}] \subseteq H'_{-v+i-1}$. Suppose it is true for $0 \leqslant i \leqslant s$. Then

$$\begin{aligned}[H_{-1}, H'_{-v+s+1}] &= \Big[H_{-1}, \sum_{0<j\leqslant s} \big[H_j, H'_{-v+s-j}\big]\Big] \\ &\subseteq \Big[\sum_{0<j\leqslant s} [H_{-1}, H_j], H'_{-v+s-j}\Big] + \Big[\sum_{0<j\leqslant s} H_j, [H_{-1}, H'_{-v+s-j}]\Big] \\ &\subseteq \sum_{0\leqslant j\leqslant s-1} [H_j, H'_{-v+s-j}] + \sum_{0<j\leqslant s} [H_j, H'_{-v+s-1-j}] \\ &\subseteq H'_{-v+s-1} + H'_{-v+s-1} \subseteq H'_{-v+s-1}\end{aligned}$$

as required.

Now it follows from g2 that

$$[H_i, H'_j] \subseteq H'_{j+i} \qquad \text{for} \quad i < 0.$$

Again by double induction on (i, j), $j \geqslant -v$, $i \geqslant 0$ beginning with $i = 0$, $j \geqslant -v$, we get

$$[H_i, H'_j] \subseteq H'_{j+1} \qquad \text{for} \quad i > 0.$$

This means that $\sum H'_j$ is an ideal of H, whence $\sum H'_j = H$, i.e., (ii) and (iii) hold in H. So the lemma is proved.

4.5.2. *The proof of Theorem* 4.1 *continued.* The algebra $S = S_i$ satisfies g1, g2, g3, g5 because of Lemmas 4.3, 4.5.1. It remains to prove that $(\operatorname{Der} S)_0$ is irreducible on S_{-1}. If it were reducible, say $V \subset S_{-1}$, $(\operatorname{Der} S)_0 V \subseteq V$, then $V \otimes B_n$ would be stable under $(\operatorname{Der} S)_0 \otimes B_n + 1_S \otimes \operatorname{Der} B_n$, and therefore (by (ii)) stable under G_0, contradiction with g4.

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