Hence

$$R_j(k) = d_h = \begin{cases} \csc(\pi h_1/k_1) & \text{if } k_1 \text{ is even} \\ \csc(\pi h_1/k_1)\cos(\pi/2k_1) & \text{if } h_1 \text{ is odd} \end{cases}$$

which proves Property 6.

It follows from this property that

$$|S_j(n)| \leqslant k/(2j+2) \quad \text{for } j \leqslant (k-2)/2$$

In fact by (4)

$$|S_j(n)| \leqslant R_j(k) < \csc(\pi h/k) \leqslant k/2h = k/(2j+2),$$

which is an improvement on (3) when $j > 0$.

In a second paper [2] we consider the sums

$$S_j(m) = \sum_{n=0}^{m} \zeta^{e(n)+jn}$$

where

$$e(n) = \sum_{i=0}^{\infty} d_i d_{i+1}$$

plays the role of $b(n)$. These sums were considered by Brillhart and Morton [1] for $k=2$. For $k>2$ they lead to infinite graphs [2] which are much more complex than those of this paper.

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DISCOVERING THEOREMS WITH A COMPUTER: THE CASE OF $y' = \sin(xy)$

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1. Introduction. The problem $y' = \sin(xy)$, $y(0) = A$, arose during an attempt to find suitable numerical examples to present to a class in differential equations and proved to be quite

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fascinating. The equation was analyzed numerically using a computer. At first the behavior of the solutions was quite baffling. They oscillated for a while (the longer the greater A) then approached zero with x tending to infinity. The conjectures describing the behavior of solutions were formulated only after the solutions for various values of A were calculated in detail.

Using geometric ideas we were then able to give a qualitative explanation of the numerical results. The additional feature which appeared during the analysis is the existence of separatrices which in the first quadrant tend to the hyperbolas $xy=2n\pi$ from below as $x\to\infty$. All other solutions tend to the hyperbolas $xy=(2n+1)\pi$ from above.

The methods of our analysis can easily be generalized to equations of the form $y'=f(g(x,y))$, where f is a function which has an infinite number of zeros without accumulation point and satisfies certain growth conditions. The function $g(x,y)$ is such that the curves $g(x,y)=c$ are concave and approach the x-axis asymptotically. There are, of course, additional conditions, connecting f and g. We chose to deal with our original $y'=\sin(xy)$ in order to leave our method as transparent as possible.

2. Properties of $y'=\sin(xy)$, $y(0)=A$. We note that the differential equation is such that the set of solution curves is symmetric with respect to the x-axis, the y-axis, and the origin. Consequently it is sufficient to consider the first quadrant. Further, Picard's theorem holds, so a unique solution passes through each point in the plane. Since $y\equiv 0$ is a solution (with $y(0)=0$), no solution satisfying $y(0)=A>0$ ever crosses the x-axis.

We refer the reader to Figure 1, which was determined numerically. We shall now attempt to verify analytically that what the figure suggests is, in fact, always true.

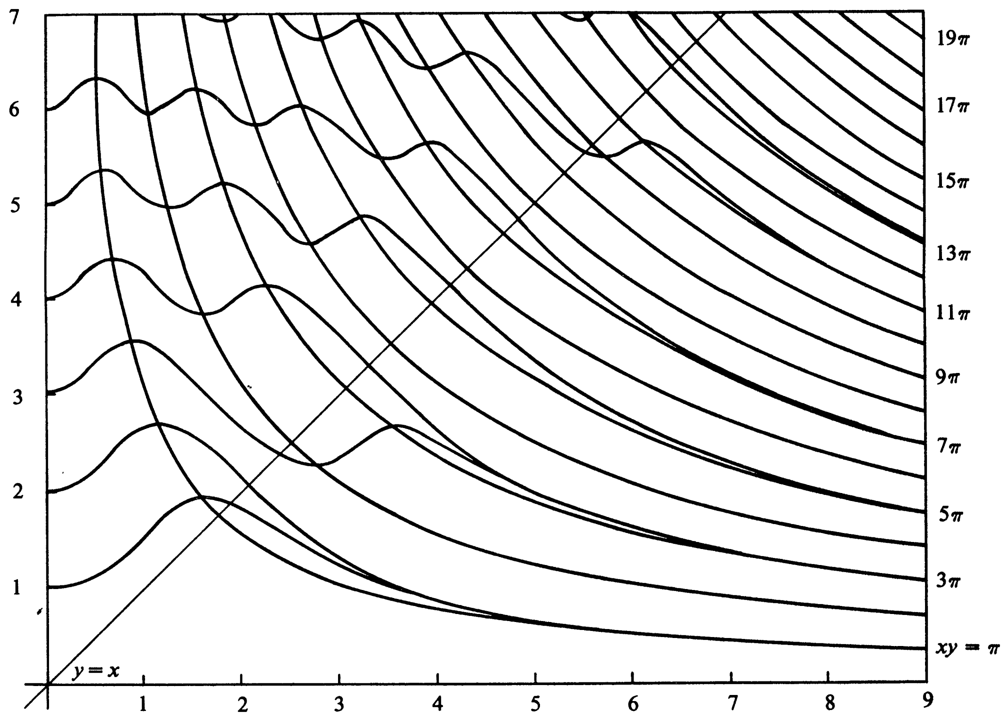

FIG. 1

LEMMA 1. *Let $y(x)$ be a solution of $y'=\sin(xy)$. Then*
(a) *If $y(x)$ intersects $xy=n\pi$, it does so with slope* 0.
(b) *If $y(x)$ intersects $xy=(n+\frac{1}{2})\pi$, it does so with slope* $(-1)^n$.

(c) $\sup|y'(x)|=1$.
(d) *If* $y(x)$ *intersects* $xy=2n\pi$, *then it also intersects* $xy=(2n+1)\pi$.

Proof. Only (d) is not completely clear. To prove (d), denote by (a,b) the intersection point of $y(x)$ and $xy=2n\pi$. Let L_1, L_2 be straight lines through (a,b) with slopes 1 and 0, respectively. Both L_1 and L_2 intersect $xy=(2n+1)\pi$ and because $0\leqslant y'(x)\leqslant 1$ for (x,y) between the curves $xy=2n\pi$ and $xy=(2n+1)\pi$, we have that $y(x)$ is below L_1 and above L_2. Thus it must intersect $xy=(2n+1)\pi$.

THEOREM 2. *Let* $y(x)$ *be a solution of* $y'=\sin(xy)$. *Then* $y(x)$ *intersects the hyperbola* $xy=\alpha$
(a) *at most once if* $2n\pi\leqslant\alpha\leqslant(2n+1)\pi$,
(b) *at most twice if* $(2n-1)\pi<\alpha<2n\pi$.

Proof. (a) Suppose (x_1,y_1), (x_2,y_2) are two intersection points. Then $f(x)=y(x)-(\alpha/x)$ satisfies $f(x_1)=f(x_2)=0$. Furthermore, $2n\pi\leqslant\alpha\leqslant(2n+1)\pi$ gives $y'(x_1)=\sin x_1y_1=\sin\alpha\geqslant 0$ and $y'(x_2)=\sin x_2y_2=\sin\alpha\geqslant 0$. Thus $f'(x_1)>0$, $f'(x_2)>0$, and by the intermediate value theorem there exists $x_3, x_1<x_3<x_2$, such that $f(x_3)=0$. Inductively, there exists a bounded sequence $\{x_i\}_1^\infty$ such that $f(x_i)=0$. Hence, there exists an accumulation point, c, of $\{x_i\}_1^\infty$, $x_1\leqslant c\leqslant x_2$. Since $f(x)$ is analytic about c, $f(x)=0$ in a neighborhood of c, a contradiction.

(b) Let $(2n-1)\pi<\alpha<2n\pi$ and let $x=c$ be the (unique) positive solution of $\sin\alpha=-\alpha/x^2$. Let $f(x)=y(x)-(\alpha/x)$. Then any intersection point, x_L, $0\leqslant x_L<c$, satisfies $f'(x_L)>0$, and any intersection point, $x_R, c<x_R<\infty$, satisfies $f'(x_R)<0$. An argument identical to that in (a) shows there is at most one intersection point in each interval $[0,c]$ and $[c,\infty]$.

COROLLARY 3. *If a hyperbola,* $xy=\alpha$, *is tangent to a solution,* $y(x)$, *then the point of tangency is unique and is the only intersection point of* $y(x)$ *and* $xy=\alpha$.

Proof. The tangency point occurs at the point, c, in the proof of Theorem 2 (b).

THEOREM 4. *For all initial values* $y(0)=A$ *the solution* $y(x)$ *to* $y'=\sin(xy)$ *intersects the line* $y=x$.

Proof. Let L be the broken line obtained as follows (Fig. 2):
1. From $(0,A)$ it has slope 1 until it intersects $xy=\pi$.
2. Between $xy=(2n-1)\pi$ and $xy=2n\pi$ it has slope 0.
3. Between $xy=2n\pi$ and $xy=(2n+1)\pi$ it has slope 1.
4. L is continuous.

Let (a_n,b_n) be the point of intersection of L with $xy=n\pi$. Then for n odd,

$$a_{n+1}=\frac{\pi(n+1)}{b_n}=a_n\left(\frac{n+1}{n}\right).$$

All that is necessary is to show L intersects $y=x$, since the solution satisfying $y(0)=A$ lies below L. This follows, since for the horizontal components of L

$$\sum_{n\,\mathrm{odd}}(a_{n+1}-a_n)=\sum_{n\,\mathrm{odd}}a_n(1/n)>a_1\sum_{n\,\mathrm{odd}}(1/n)=\infty.$$

So while the diagonal components parallel $y=x$, the horizontal components push it relatively ever farther to the right and ultimately beyond.

Until the solution intersects $y=x$, it crosses the regions between the hyperbolas $xy=n\pi$. The slopes of y within these regions are alternately positive and negative giving

THEOREM 5. *Until a solution* $y(x)$ *crosses* $y=x$, *it is alternately increasing and decreasing. The solution oscillates.*

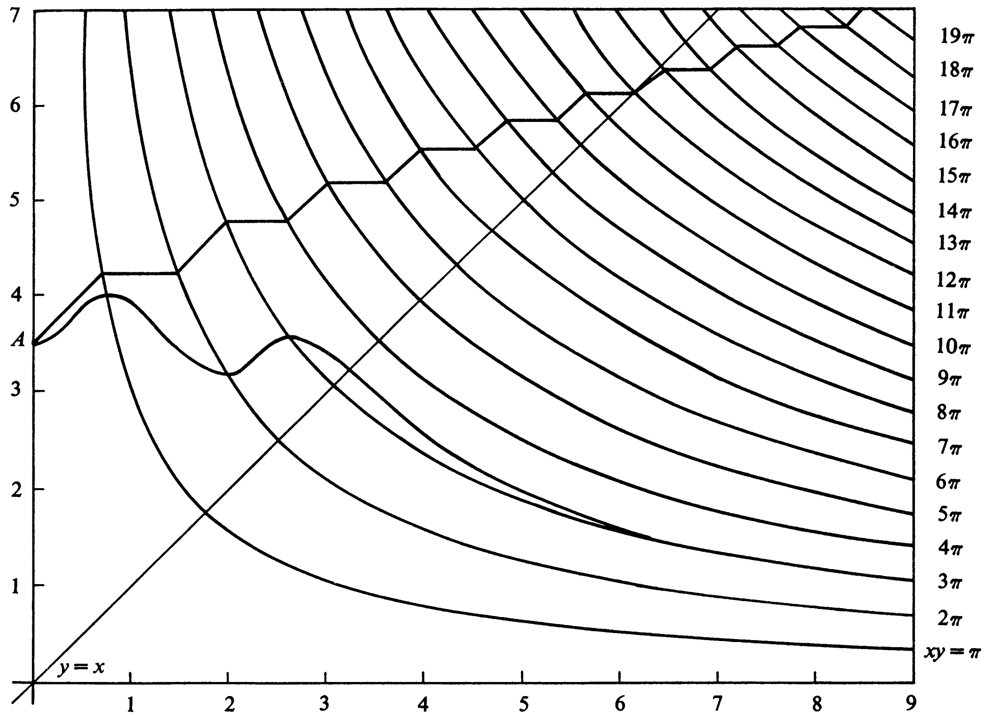

FIG. 2

THEOREM 6. *Let a solution $y(x)$ intersect $y=x$ at $x=x_0$. Let $x_0y(x_0)<(2n+\frac{3}{2})\pi$. Then the solution $y(x)$ does not intersect $xy=(2n+\frac{3}{2})\pi$.*

Proof. First note that once the solution lies below $y=x$, it remains below by Lemma 1(c). Assume the conclusion is false. Let (a,b) be the first point of intersection with $xy=(2n+\frac{3}{2})\pi$ below $y=x$. Such a first point exists by Theorem 2. The slope $y'(x)$ at a is -1, while the slope of the hyperbola is greater than -1. This implies that to the left of a the solution is above the hyperbola. This further implies that either the solution and the hyperbola intersect to the left of a and below $y=x$ or $x_0y(x_0)>(2n+\frac{3}{2})\pi$. Both are impossible.

Likewise, if the solution intersects $xy=(2n+\frac{3}{2})\pi$ above $y=x$, then in order to pass through $(x_0,y(x_0))$ it would have to intersect $xy=(2n+\frac{3}{2})\pi$ again above $y=x$. Again slope considerations make this impossible.

COROLLARY 7. *For each solution $y(x)$, there exists a maximum n such that the solution intersects $xy=(2n+1)\pi$. The solution lies between $xy=(2n+1)\pi$ and $xy=(2n+2)\pi$ for all sufficiently large x.*

Proof. In the open region bounded by the x-axis, the y-axis, and $xy=\pi$, $0<y'<1$. So $y(x)$ is increasing and is bounded above by $y=A+x$, below by $y=A$. Since both intersect $xy=\pi$, $y(x)$ does so as well.

The existence of such an n is now guaranteed by Theorem 6.

Once $y(x)$ intersects $xy=(2n+1)\pi$, it cannot do so again according to Theorem 2. By Lemma 1(d) it cannot intersect $xy=(2n+2)\pi$. Therefore it remains between $xy=(2n+1)\pi$ and $xy=(2n+2)\pi$.

3. The Asymptotic Nature of Solutions. In order to adequately describe the asymptotic nature of the solutions as $x\to\infty$, let us consider the following regions (see Fig. 3): Let

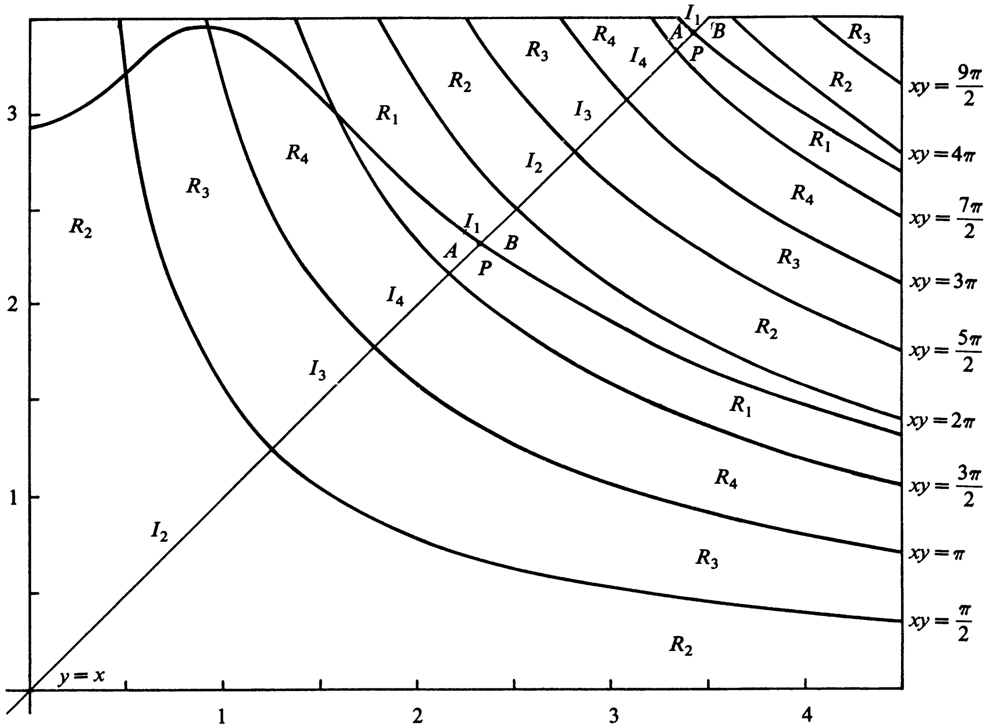

FIG. 3

$$
\begin{aligned}
R_1&=\left\{(x,y):\left(2n-\tfrac{1}{2}\right)\pi\leqslant xy\leqslant 2n\pi\right\},\\
R_2&=\left\{(x,y):2n\pi\leqslant xy\leqslant\left(2n+\tfrac{1}{2}\right)\pi\right\},\\
R_3&=\left\{(x,y):\left(2n+\tfrac{1}{2}\right)\pi\leqslant xy\leqslant(2n+1)\pi\right\},\\
R_4&=\left\{(x,y):(2n+1)\pi\leqslant xy\leqslant\left(2n+\tfrac{3}{2}\right)\pi\right\},\\
R'_1&=\left\{(x,y):\left(2n+\tfrac{3}{2}\right)\pi\leqslant xy\leqslant(2n+2)\pi\right\}.
\end{aligned}
$$

Let

$$
\begin{aligned}
I_1&=R_1\cap\{\text{line } y=x\},\\
I_2&=R_2\cap\{\text{line } y=x\},\\
I_3&=R_3\cap\{\text{line } y=x\},\\
I_4&=R_4\cap\{\text{line } y=x\},\\
I_1'&=R_1'\cap\{\text{line } y=x\}.
\end{aligned}
$$

We shall examine in succession the solutions $y(x)$ which pass through I_1, I_2, I_3, I_4, I_1'.

I_1: The solution passing through the point $\left(\sqrt{\left(2n-\frac{1}{2}\right)\pi}\ ,\ \sqrt{\left(2n-\frac{1}{2}\right)\pi}\ \right)$, the left end of I_1, must drop below $xy=(2n-\frac{1}{2})\pi$, since its slope at that point is -1. By Corollary 3 any other passage through $xy=(2n-\frac{1}{2})\pi$ is impossible. Likewise, solutions intersecting I_1 near the left end of I_1 must intersect $xy=(2n-\frac{1}{2})\pi$ below $y=x$, leaving R_1 to remain in the region below, by Theorem 2. We shall show that these solutions become asymptotic to $xy=(2n-1)\pi$ when we consider the regions I_1' and R_4.

Similarly the solution passing through the point $(\sqrt{2n\pi}\ ,\ \sqrt{2n\pi}\ )$, the right end of I_1, must pass into R_2 and never return to R_1. Likewise, solutions intersecting I_1 near the right end of I_1 must intersect $xy=2n\pi$ below $y=x$, leaving R_1, and remaining above.

Thus there exist three sets contained in I_1,

$A=\{(x,x)$: the solution passing through (x,x) intersects $xy=(2n-\frac{1}{2})\pi$ below $y=x$ and remains below $xy=(2n-\frac{1}{2})\pi\}$,

$B=\{(x,x)$: the solution passing through (x,x) does not intersect $xy=(2n-\frac{1}{2})\pi$ or $xy=2n\pi$ below $y=x\}$,

$C=\{(x,x)$: the solution passing through (x,x) intersects $xy=2n\pi$ below $y=x$ and remains above $xy=2n\pi\}$.

A moment's reflection establishes that A and C are nonempty intervals

$$\left(\sqrt{(2n-\tfrac{1}{2})\pi}\,,\sqrt{(2n-\tfrac{1}{2})\pi}\,\right)\in A,\ \ (\sqrt{2n\pi}\,,\sqrt{2n\pi}\,)\in C$$

with B in between. The boundary points between A and B, B and C are neither in A nor in C, since if the boundary p between A and B were in A, then points above p would be in A. If the boundary p between B and C were in C, then points below p would be in C. B is nonempty since there cannot be a last point of A or a first point of C.

THEOREM 8. (a) *Solutions passing through A in I_1 intersect $xy=(2n-\frac{1}{2})\pi$ and remain below, becoming asymptotic to $xy=(2n-1)\pi$.*

(b) *The set B consists of exactly one point p_n. The solution passing through p_n remains in I_1. This solution $y(x)$ becomes asymptotic to $xy=2n\pi$, and $xy(x)\to 2n\pi$.*

(c) *Solutions passing through C in I_1 intersect $xy=2n\pi, xy=(2n+\frac{1}{2})\pi$ and $xy=(2n+1)\pi$, passing through R_2, through R_3, into R_4, where they become asymptotic to $xy=(2n+1)\pi$.*

Proof. (a) We shall establish the asymptotic nature of the solutions when R_4 is examined in detail.

(b) Let $y(x)$ be a solution passing through B.

If $xy(x)$ does not approach $2n\pi$, then by Theorem 2 it must ultimately be bounded away from $2n\pi$. In that case there is an $\varepsilon>0$ such that $\sin(xy)<-\varepsilon<0$. This implies that $y'<-\varepsilon$, and forces $y(x)$ to intersect $xy=(2n-\frac{1}{2})\pi$, which is contrary to assumption.

It is apparent that B is closed. Let $y(x)$ represent the solution passing through the left end of B, and let $Y(x)$ be any other solution passing through B. Then $Y(x)>y(x)$, and $xY(x)$ and $xy(x)$ both approach $2n\pi$. Thus

$$Y'-y'=\sin(xY)-\sin(xy)=\int_{xy}^{xY}\cos t\,dt.$$

Since xY and xy are eventually close to $2n\pi$, there is a $\delta>0$ such that $\cos t>1-\delta>0$, for $xy<t<xY$, x sufficiently large. Hence,

$$Y'-y'>\int_{xy}^{xY}(1-\delta)\,dt,$$

$$=x(1-\delta)(Y-y).$$

This implies for some $C>0$,

$$Y-y>C\exp\left[(1-\delta)x^2/2\right]$$

and $Y\to\infty$ as $x\to\infty$. This is impossible, so B contains only one point.

(c) We shall establish the asymptotic nature of the solutions when R_4 is examined in detail.

I_2: Solutions passing through I_2 have positive slope at that point. According to Corollary 7, they must intersect $xy=(2n+1)\pi$ and enter R_4. By Theorems 2 and 6 they must remain in R_4.

THEOREM 9. *Solutions passing through I_2 intersect $xy=(2n+\frac{1}{2})\pi$ and $xy=(2n+1)\pi$, passing through R_3 into R_4, where they remain in R_4 and become asymptotic to $xy=(2n+1)\pi$.*

Proof. The asymptotic nature of the solutions will be established when R_4 is examined in detail.

I_3: Solutions passing through I_3 also have positive slope and must intersect $xy=(2n+1)\pi$ and enter R_4 by Corollary 7. They must remain in R_4 by Theorems 2 and 6.

THEOREM 10. *Solutions passing through I_3 intersect $xy=(2n+1)\pi$ passing into and remaining in R_4, where they become asymptotic to $xy=(2n+1)\pi$.*

I_4: According to Theorem 2 solutions passing through I_4 must remain above $xy=(2n+1)\pi$. According to Theorem 6 they must remain below $xy=(2n+\frac{3}{2})\pi$. Thus they remain in R_4.

THEOREM 11. *Solutions passing through I_4 remain in R_4 and become asymptotic to $xy=(2n+1)\pi$.*

I_1': Solutions passing through I_1' divide themselves into three classes, just as those passing through I_1. Those which are of interest to us intersect $xy=(2n+\frac{3}{2})\pi$ below $y=x$ and remain in R_4.

THEOREM 12. *Solutions passing through I_1' within the interval A of that region intersect $xy=(2n+\frac{3}{2})\pi$, remain in R_4 and become asymptotic to $xy=(2n+1)\pi$.*

The behavior in R_4: We have established that only the solution passing through p_n, which becomes asymptotic to $xy=2n\pi$, fails to enter R_4 (or its counterpart, associated with the integer $n-1$).

THEOREM 13. *All solutions passing through a point in R_4 below $y=x$ remain in R_4 and become asymptotic to $xy=(2n+1)\pi$. For such solutions $xy(x)\to(2n+1)\pi$ as $x\to\infty$.*

Proof. If, for a solution $y(x)$, $xy(x)$ did not approach $(2n+1)\pi$, then by Theorem 2 there exists an $\varepsilon>0$ such that $\sin xy<-\varepsilon<0$. This immediately forces an intersection of such a solution with $xy=(2n+1)\pi$, which is impossible.

MAIN THEOREM. *Let $y(x)$ be a solution of $y'=\sin(xy)$, $y(0)=A$, $A>0$, in the first quadrant.*

(a) *$y(x)$ intersects the line $x=y$ at some point (a,a). It oscillates until it intersects this line.*

(b) *If $(a,a)=p_n$ of Theorem 8, then $y(x)$ approaches the hyperbola $xy=2n\pi$ asymptotically from below and $x\cdot y(x)\to 2n\pi$ as $x\to\infty$.*

(c) *If (a,a) lies between p_{n-1} and p_n, then $y(x)$ approaches the hyperbola $xy=(2n-1)\pi$ from above, and $x\cdot y(x)\to(2n-1)\pi$ as $x\to\infty$. Moreover $y(x)$ intersects the hyperbola $xy=(2n-\frac{1}{2}\pi$ exactly once if $(a,a)=\left(\sqrt{(2n-\frac{1}{2})\pi}\,,\sqrt{(2n-\frac{1}{2})\pi}\right)$ and exactly twice if (a,a) is between $\left(\sqrt{(2n-\frac{1}{2})\pi}\,,\sqrt{(2n-\frac{1}{2})\pi}\right)$ and p_n.*

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