One-Dimensional Affine Group Schemes*

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Our objects of study are affine group schemes, finitely presented and flat, over a domain $A$. As in [10], we call such group schemes models of their generic fibers. In dimension one there are only a few possibilities for the generic fiber, and we are able to obtain a complete classification in some generality:

THEOREM. Let $A$ be an integrally closed domain with perfect fraction field. Then the smooth group schemes over $A$ with connected affine one-dimensional fibers correspond to rank 2 projective algebras over $A$.

The construction of these groups is straightforward; to an algebra $B$ we assign the quotient $R_{R/A}G_{m}/G_{m}$. The main effort comes in showing that these are the only possibilities. The key to this, and the basic technical idea in the paper, is the use of Néron blow-ups of group schemes over valuation rings. This process has been used before [1, 2] as a tool for resolving singularities, but in fact it also furnishes a good grasp on the structure of models in general. A further instance of this is found in the last section, where we give an analysis of all models (smooth or not) of $G_{m}$ over a valuation ring. It seems likely that the technique should also have other applications.

1. NÉRON BLOW-UPS

Let $A$ be a discrete valuation ring with fraction field $K$, uniformizer $\pi$, and residue field $k$. If $G$ is a flat separated group scheme of finite type over $A$, and the generic fiber $G_{k}$ is affine, then $G$ itself is affine [1, p. 30]; throughout this section $G$ will be assumed to be such a model. Our goal here is to acquire information about the blow-up process for passing from $G$ to some

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other model of $G_K$. Propositions 1.1 and 1.2 are already on record in more generality [1], but we write them out for completeness because they are so simple. The most important result for the later applications is Theorem 1.4 together with Corollary 1.8.

**Construction and Universal Property**

Let $G$ be an affine model. By flatness we can view the ring of global sections $A[G]$ as an $A$-subalgebra of $K[G]$. Suppose $H$ is a closed subgroup of the special fiber $G_\mathrm{sp}$, defined by some proper ideal $J$ of $A[G]$ containing $\pi$. Let $G''$ be $\text{Spec } A[\pi^{-1}J]$, where $A[\pi^{-1}J]$ is taken as a subalgebra of $K[G]$. This $G''$ is the Néron blow-up of $H$ in $G$.

**Proposition 1.1.** $G''$ is a model of $G_K$.

**Proof.** Since $\pi A[G] \subseteq J$, we have $A[G] \subseteq A[\pi^{-1}J] = A[G'']$; clearly $K[G''] = K[G]$. The ideal $J$ is finitely generated, since $A[G]$ is noetherian; and if $J = (\pi, f_1, \ldots, f_n)$, then $A[G''] = A[G][\pi^{-1}f_1, \ldots, \pi^{-1}f_n]$, and in particular $G''$ is again of finite type. It is flat since by construction $A[G'']$ is torsion-free. As $H$ is a subgroup, the comultiplication $\Delta$ sends $J$ into $J \otimes A[G] + A[G] \otimes J$, so $\Delta(\pi^{-1}J) \subseteq A[G''] \otimes A[G'']$; similarly the antipode preserves $A[G'']$, and the counit $\varepsilon$ maps it into $A$ since $\varepsilon(J) \equiv 0 \pmod{\pi}$.

For brevity we call a homomorphism $G' \to G$ a model map if it is an isomorphism on the generic fibers.

**Proposition 1.2.** The canonical model map $G'' \to G$ sends the special fiber into $H$, and any model map $G' \to G$ sending the special fiber into $H$ factors uniquely through $G''$.


**Remark.** The group $G''(A)$ is canonically identified with those points in $G(A)$ whose reductions modulo $\pi$ are in $H$. Thus $G''$ can be thought of as the group scheme corresponding to a congruence subgroup of $G$.

**Standard Blow-up Sequence**

**Lemma 1.3.** A model map that is epimorphic on the special fiber is an isomorphism.

is, $A[G] \cap \pi A[G'] = \pi A[G]$. Since the modules are torsion-free with the same $K$-span, this makes them equal.  

**Theorem 1.4.** Any model map is (isomorphic to) a composite of blow-ups.

**Proof.** Again we take $A[G] \subseteq A[G']$. If the (closed) image $H$ of $G'$ is all of $G_k$, we have $G' = G$ by the lemma; if not, then $G' \to G$ factors through $G_i = G^H$. If $G'_k$ does not map onto $(G_i)_k$, we blow up its image to get $G' \to G_2$, and so on. We must show this process eventually stops. But the ideal $J$ defining $H$ is $A[G] \cap \pi A[G']$, so $A[G_i]$ contains $\pi^{-1}J = A[G'] \cap \pi^{-1}A[G]$. Then $A[G_2]$ contains $A[G'] \cap \pi^{-2}A[G]$, and in general $A[G_n]$ contains $A[G'] \cap \pi^{-n}A[G]$. Since $K[G'] = K[G]$, we have $A[G'] \subseteq \bigcup A[G_n]$, and equality occurs at some finite stage because $A[G']$ is finitely generated.  

**Remarks.** (1) We call the sequence $G_i$ constructed in the proof the standard blow-up sequence from $G$ to $G'$. At every stage in it the subgroup blown up maps onto the subgroup blown up at the previous stage, since these subgroups are at each stage just the image of $G_k$. Conversely it is easy to see inductively that a sequence of blow-ups with this property is the standard sequence between its beginning and its end.

(2) For blow-ups in general, $G^H_k$ need not map onto $H$. For example, let char$(K) = 2$, and let $G$ be the model of $G_a$ given by $A[x, y]$ where $\Delta x = x \otimes 1 + 1 \otimes x$ and $\pi y = x^2$. Over $k$ we have $x^2 \equiv 0$ and $x$ and $y$ primitive, so $G_k \simeq G_a \times \alpha_2$. The subgroup $H = G_a$ is defined by $x \equiv 0$, so $A[H^H] = A[x, y, \pi^{-1}x]$. Here $x = \pi(\pi^{-1}x)$ and $y = \pi(\pi^{-1}x)^2$, so in $k[G] \to k[G^H]$ both $x$ and $y$ go to zero, and thus $G^H_k$ has trivial image in $G_k$.

(3) The theorem does not imply the ascending chain condition on models between $G$ and $G'$. For example, let char$(K) = 2$, let $G = G_a = \text{Spec } A[x]$, and set

$$A[G_1] = A[x, \pi^{-1}x^2],$$

$$A[G_2] = A[x, \pi^{-1}x^2, \pi^{-3}x^4],$$

$$A[G_3] = A[x, \pi^{-1}x^2, \pi^{-3}x^4, \pi^{-7}x^8],$$

and so on. It is easy to see that the $G_i$ are models, and in fact $G, G_1, G_2, \ldots, G_i$ is the standard blow-up sequence from $G$ to $G_i$. But all $A[G_i]$ are contained in $A[G^{\text{spec}}] = A[\pi^{-1}x]$.

(4) The group structure has played no essential role in the hypotheses or proofs so far.
Kernel of the Blow-Up

**Theorem 1.5.** Let $H$ be a subgroup of $G_k$ defined by an ideal $J$ in $A[G]$, let $I$ be the augmentation ideal of $A[G]$, and let $r = \dim_k(J/\pi A[G] + IJ)$. Then the kernel of $G^H_k \to G_k$ is isomorphic to a subgroup of $G'_o$.

**Proof.** Let $F$ be the kernel. The ring $A[G][H]$ is generated over $A$ by $\pi^{-1}J$, so $k[F] = A[G][H]/(\pi, I) A[G][H]$ is generated by the images of the $\pi^{-1}x$ for $x$ in $J$. Clearly such an image is constant for $x$ in $A[G]$ and zero for $x$ in $IJ$. Thus $k[F]$ is generated by $r$ elements. We must show now that $\Delta \pi^{-1}x$ is congruent to $\pi^{-1}x \otimes 1 + 1 \otimes \pi^{-1}x$ for each $x$ in $J \cap I$. As $H$ is a subgroup, there is some expression $A \Delta x = x \otimes 1 + 1 \otimes x + \sum y_i \otimes z_i$, where in each term either $y_i$ or $z_i$ is in $J$. The group axiom $(1 \otimes \varepsilon) \circ A = \varepsilon$ (existence of identity element) gives us $\sum y_i \otimes \varepsilon(z_i) + 1 \otimes \varepsilon(x) = 0$. As $x \in I$, it follows that $\sum y_i \otimes \varepsilon(z_i) = 0$; and if $z_i \in J$, then $\varepsilon(z_i) \in \pi A \subset J$. Thus we can write $A \Delta x = x \otimes 1 + 1 \otimes x + \sum y_i \otimes (z_i - \varepsilon(z_i))$. Repeating this on the other side, we may assume $y_i, z_i \in I$. Now $A \Delta \pi^{-1}x = (\pi^{-1}x) \otimes 1 + 1 \otimes (\pi^{-1}x) + \sum \pi^{-1}(y_i \otimes z_i)$, and the result follows since $\pi^{-1}J \otimes I$ and $I \otimes \pi^{-1}J$ go to zero in $k[F] \otimes k[F]$.

**Corollary 1.6.** If $H$ contains the connected component of $G_k$, the kernel is trivial.


**Remark.** The kernel need not be a full vector group, and may even be disconnected. For example, let $\text{char}(K) = 2$, and let $A[G] = A[x, y]$ where $x$ and $y$ are primitive and $\pi y = y^2 + x^4$. (This is a model of a twisted form of $G_a$.) We get $G^{\text{ref}}$ by adjoining $u = \pi^{-1}x$ and $v = \pi^{-1}y$, so $A[G^{\text{ref}}] = A[u, v]$; here $v = v^2 + \pi^2 u^4$, and it is easy to see this is a defining equation. Thus over $k$ the kernel, which of course is all of $G^{\text{ref}}_k$, is $\mathbb{Z}/2\mathbb{Z} \times G_a$.

**Smooth Connected Case**

**Theorem 1.7.** Let $G$ be a smooth model with connected fibers. Let $H$ be a smooth connected subgroup of $G_k$. Then $G^H_k$ is smooth with connected fibers, the special fiber maps onto $H$, and the kernel is a vector group whose dimension is the $r$ in Theorem (1.5).

**Proof.** As $H$ is smooth and connected, it has no proper subgroups of the same dimension; thus the image either is $H$ or has lower dimension. In the notation of the last theorem, the kernel either equals $G'_o$ or has lower dimension. By flatness [6, VI, 4.3] we have $\dim G^H_k = \dim G^H_k = \dim G_k = \dim G_k$, and thus $\dim(\text{kernel}) + \dim(\text{image}) = \dim G_k$. By smoothness,
dim $G_k = \dim_k(I/\pi I + I^2)$ and $\dim H = \dim_k(I/\pi I^2_H)$. If $\Omega^1_{G_\alpha}$ denotes the module of differentials on $G$, then also by smoothness there is a sequence of $k[\!G\!]$-modules

$$0 \to J/\pi A[G] + J^2 \to \Omega^1_{G_\alpha}/(\pi A[G] + J) \Omega^1_{G_I} \to \Omega^1_{H_I} \to 0$$

which is exact and split [5, O_{IV}, §20]. Dividing by $I$, we get an exact sequence

$$0 \to J/\pi A[G] + IJ \to I/\pi I + I^2 \to I/I^2_H \to 0.$$ 

Thus $r + \dim H = \dim G_k$. Hence we must have equality of groups in both the kernel and the image. Then $G_H''$ is smooth and connected since the kernel and image are, and $G''_H$ is smooth because it is flat with smooth fibers [5, 17.5.2].

One can see also how the action of $H$ here on $G'_\alpha$ is constructed from the adjoint representation of $G_k$.

**Corollary 1.8.** Let $G' \to G$ be a model map of smooth models with connected fibers. Then each stage in the standard blow-up sequence from $G$ to $G'$ is smooth with connected fibers.

**Proof:** At each stage the image of $G'_k$ is smooth and connected, since $G'_k$ itself is so. Induction using the theorem shows then that each stage is smooth with connected fibers.

### 2. Smooth One-Dimensional Groups over Valuation Rings

Over our valuation ring $A$ we are now able to classify all smooth affine $G$ with connected one-dimensional fibers. We will see for instance that $G_m$ is an absolutely minimal model, i.e., $A[G_m]$ is actually contained in every other Hopf subalgebra spanning $K[G_m]$; hence we get all models from $G_m$ by a sequence of blow-ups. Models of other tori can then be treated by descent.

For the additive group we have at least that $G_a$ is a relatively minimal model, i.e., every model $G$ has a model map to $G_a$; thus the same method works, and again from that we get the twisted additive groups.

**Models of $G_a$**

**Lemma 2.1.** The additive group $G_a$ is a relatively minimal model.

**Proof.** Let $G$ be any model. Then $K[G] = K[x]$ with $x$ primitive. Changing $x$ by a scalar multiple, we may suppose $x$ is in $A[G]$. Then $G$ has a model map to $G_a = \text{Spec } A[x]$. 


Theorem 2.2. Any model of the additive group which is smooth with connected fibers is isomorphic to $G_a$.

Proof. Let $G$ be any such model. By the lemma, $G$ maps to $G_a$. The image of the special fiber is smooth and connected, so is trivial or all of $(G_a)_k$. In the second case $G = G_a$ by Lemma 1.3. Otherwise, we blow up the trivial subgroup; this gives us $A[x, \pi^{-1}x] = A[\pi^{-1}x]$, which is just another copy of $G_a$. Continuing this process, we get $G_a$ at every stage, and by Theorem 1.4 one of these stages eventually is $G$. \qed

This result was proved by a quite different method in [10].

Models of $G_m$

Proposition 2.3. Let $G$ be a flat affine group over $A$, and let $t$ be a character of $G_k$. Then $t$ is in $A[G]$.

Proof. We may assume $t \neq 1$. Let $M = (K \cdot 1 + K \cdot t) \cap A[G]$, and write $y = t - 1$. Since $\varepsilon(t) = 1$ and $\varepsilon(A[G]) \subseteq A$, we have $M = A \cdot 1 + J \cdot y$ for some $A$-submodule $J$ of $K$; $J$ is nonzero, since $A[G]$ spans $K[G]$. For any $a$ in $J$, the element $\Delta(ay) = ay \otimes 1 + 1 \otimes ay + a(y \otimes y)$ must lie in $M \otimes M$, and hence $a \in J \cdot J$. But clearly $J \subseteq J^2$ forces $1$ to be in $J$. \qed

Corollary 2.4. A diagonalizable group is an absolutely minimal model.

Proof. The characters span $A[G]$. \qed

Theorem 2.5. Let $G$ be smooth with connected fibers, $G_k = G_m = \text{Spec } K[t, t^{-1}]$. Then $A[G]$ is $A[t^{-1}, \pi^{-n}(t - 1)]$ for some $n \geq 0$.

Proof. By the corollary, $G$ has a model map to $G_m$. As in Theorem 2.2, if this is not an isomorphism, the smooth connected image of the special fiber must be the trivial subgroup, which is defined by $t \equiv 1$. Then $G$ maps to $G_1 = \text{Spec } A[t^{-1}, \pi^{-1}(t - 1)]$. The special fiber of $G_1$ is $G_a$, with coordinate the reduction of $\pi^{-1}(t - 1)$. If $G_k$ does not map onto this, the image is again trivial, and $G$ maps to $G_2 = \text{Spec } A[t^{-1}, \pi^{-2}(t - 1)]$. This in turn has special fiber $G_a$, and the construction continues until $G$ is reached. \qed

In Section 4 we will refine this argument to obtain an analysis of all models of $G_m$. The same argument also applies to models of $\mu_a$: the sequence starts with $\text{Spec } A[t]$ where $t^p = 1$ and yields $\text{Spec } A[\pi^{-n}(t - 1)]$ as the other models. Except when char $K = p$, however, this sequence stabilizes; for when $n(p - 1) > \text{ord}_s(p)$, the ring is no longer a free $A$-module, and the special fiber is trivial.
Models of Twisted Tori

**Theorem 2.6.** Let $G$ be smooth with connected fibers, and suppose $G_k$ is a one-dimensional anisotropic torus. Then $G$ has the form $\text{Spec } A[v, w]$ where $v = w^2 + r vw + sv^2$ and

$$
\begin{align*}
\Delta v &= v \otimes 1 + 1 \otimes v + (r^2 - 2s) v \otimes v + 2w \otimes w \\
&\quad + rv \otimes w + rw \otimes v, \\
\Delta w &= w \otimes 1 + 1 \otimes w - rs v \otimes w - rw \otimes w \\
&\quad - 2sv \otimes w - 2sw \otimes v.
\end{align*}
$$

**Proof.** There is a separable quadratic extension $K'$ of $K$ with $G_{K'} \simeq G_m$. Let $A'$ be the integral closure of $A$ in $K'$, so $A'$ is a principal ideal domain (with at most two primes). If $A'$ is a discrete valuation ring, we know from Theorem 2.5 that $A'[G]$ has the form $A'[t^{-1}, b^{-1}(t - 1)]$ for some $b$. But even if there are two primes, the multiples of $t - 1$ in $A'[G]$ have bounded denominators, since they do after each localization; hence they equal the multiples of some $b^{-1}(t - 1)$. Then here again $A'[t^{-1}, b^{-1}(t - 1)]$ is the whole of $A'[G]$, since it is so after localization.

**Lemma 2.7.** The elements $v_m = t^{-1m/2}(b^{-1}(t - 1))^m$ for $m = 0, 1, 2,...$ form an $A'$-basis of $A'[G]$.

**Proof.** When we expand the $v_m$ in powers of $t$, we see each involves a new power, so they are independent. Clearly they lie in $A'[G]$. One computes that $v_mv_n = v_{m+n}$ unless $m$ and $n$ are both odd, in which case $v_mv_n = v_{m+n} + bv_{m+n+1}$; thus their $A'$ span is an algebra. It contains $1 = v_0$ and $b^{-1}(t - 1) = v_1$ and $t^{-1} = b^2v_2 - bv_1 + v_0$, so it is all of $A'[G]$. 

Let $\sigma$ be the conjugation of $K'$ over $K$. By assumption, $G_k$ is not split, so we must have $\sigma(t) = t^{-1}$ in $K[G] \otimes_k K' = K'[t, t^{-1}]$. Furthermore, since $A'$ is faithfully flat over $A$, the algebra $A[G]$ must be free. In $A'[G] = A[G] \otimes A'$, then, the set of $\sigma$-fixed elements is $A[G]$. In the basis $\{v_m\}$ we compute that $\sigma(v_m)$ is $(-b/\sigma(b))^m v_m$ for $m$ even and $(-b/\sigma(b))^m (v_m - bv_{m+1})$ for $m$ odd. Since the $\sigma$-fixed elements of $A'[G]$ span $A'[G]$, clearly those in each $A'v_{2m-1} + A'v_{2m}$ must span $A'v_{2m-1} + A'v_{2m}$. This tells us first of all that $b/\sigma(b)$ is invertible, but we can go on to deduce a stronger statement. Consider the case $m = 1$. The element $v = (-b/\sigma(b))v_1$ is fixed. Another element $cv_1 + av$ then will be fixed iff $c = \sigma(c)(-b/\sigma(b))$ and $a = \sigma(a) - \sigma(c)b$. If this gives us a basis of fixed elements, then $c$ must be invertible in $A'$. Now our original element $b$ was only determined up to invertible elements, so we may go back and replace $b$ by $\sigma(c)b = \sigma(a) - a$. 


We now compute fixed elements in $A'[G]$ for $b$ of the form $\sigma(a) - a$. Then $-b/\sigma(b) = 1$, so $v = v_2 = b^{-2}[t^{-1} - 2 - t]$ and $w = v_1 + av_2 = b^{-1}(t - 1) + av$ are fixed. Since $v' = v_{2m}$ and $v' - w = v_{2m - 1} + av_{2m}$, we see that the fixed algebra is precisely $A[v, w]$. From the formulas for $v$ and $w$ it is straightforward to compute $\Delta v$, $\Delta w$, and the relation; the constants occurring are $r = -(a + \sigma(a))$ and $s = a\sigma(a)$.

Remarks. (1) The proof shows that $\text{Spec} A'[t^{-1}, b^{-1}(t - 1)]$ arises by base-extension from $A$ iff $b$ is a unit times an element $\sigma(a) - a$. If, for instance, $A'$ is a valuation ring unramified over $A$, this puts no restriction on $b$. But if $A'$ is tamely ramified (char $k \neq 2$), it means that $b$ has odd order. The models over $A'$ arising in this way are here the complement of those arising by extending an $A$-model of $G_m$.

(2) Since $b \neq 0$, we have $K' = K(a)$, and thus the last sentence in the proof shows that the quadratic form $a^2 + mab + n\beta^2$ is the norm form from $K'$ to $K$.

(3) The true significance of this group law in $v$ and $w$ will be explained in the next section, where we will extend the result to other base rings.

Models of Twisted Additive Groups

**Theorem 2.8.** Suppose the generic characteristic is $p$. The smooth affine groups with connected fibers and generic fiber a form of the additive group are the subgroups of $G_a^2$ defined by equations of the form

$$y^{p^n} = x + a_1x^{p^n} + \cdots + a_mx^{mp^n}.$$ 

**Proof.** We know by [9] that the group over $K$ has such an equation with $a_i$ in $K$. Replacing $y$ by $\pi^{-t}y$ and $x$ by $\pi^{-t}\pi^n x$ for large $s$, we may assume the $a_i$ are in $A$. Then $G = \text{Spec} A[x, y]$ is one model of the correct form. Let $K' = K(\pi^{p^{-n}}, a_1^{p^{-n}}, \ldots, a_m^{p^{-n}})$, and let $A'$ be the integral closure of $A$ in $K'$, so $A'$ is a valuation ring with uniformizer $\lambda = \pi^{p^{-n}}$. We have $A' \otimes A[G] = A'[t]$ with $t^n = x$.

Suppose now $R \subseteq K[x, y]$ represents another model smooth with connected fibers. By Theorem 2.2 then $A' \otimes R = A'[\lambda^r t]$ for some integer $r$. Let $x_1$ be $\pi^r x$, which is in $A' \otimes R$ and hence in $R$. The primitive elements in $A' \otimes R$ are the $A'$-span of those in $R$, since $A \rightarrow A'$ is faithfully flat, and hence $R$ contains a primitive $y_1$ which begins with $\lambda^r t$. Then $y_1$ has degree $p^n$ over $K'(x_1) = K'(t^{p^n})$, so $K(x_1, y_1)$ must be all of $K(x, y)$. As $y_1$ is a $p$-polynomial in $t$, its equation is $y_1^{p^n} = x_1 + b_1x_1^{p^n} + \cdots + b_rx_1^{mp^n}$ for some $b_i$ in $K$. Since $A'[\lambda^r t]$ is integral over $A[x_1]$, the $b_i$ are actually in $A$. The form of the
equation shows now that \( A[x_1, y_1] \) is integrally closed. But \( R \) is integral over it and has the same fraction field. Thus \( R = A[x_1, y_1] \).

**Corollary 2.9.** Let \( G \) be smooth with connected one-dimensional fibers, and suppose \( G_k \) is a rational affine curve. Then \( G \) has the form described in Theorem 2.6 for some \( v \) and \( w \).

**Proof:** The group must be \( G_a \) or \( G_m \) over the algebraic closure [3, p. 257]. Looking at the equations of twisted forms, one sees that the rational \( G_k \) are tori and \( G_a \) and the groups \( y^2 = x + cx^2 \) in characteristic 2. (We are grateful to Russell for pointing this out.) For \( y^2 = x + cx^2 \), take \( v = x \) and \( w = y \); for \( G_a \), take \( w = x \) and \( v = x^2 \); for \( A[t, b'(t-1)] \), take \( w = b(t-1) \) and \( v = b^2(t^{-1}-2t) \); for anisotropic \( G_k \), use Theorem 2.6.

3. **Global Classification of Smooth One-Dimensional Groups**

In this section we obtain in fair generality a classification of smooth affine group schemes with connected one-dimensional fibers. First we construct a collection of such groups explicitly. Then for an arbitrary model \( G \) we use the results of the previous section to produce one of our collection which agrees with \( G \) over all discrete valuation rings. A theorem of Raynaud then shows the two are actually isomorphic.

**Construction of the Groups**

**Theorem 3.1.** Let \( A \) be a ring, and let \( B \) be an \( A \)-algebra with unit which is a projective rank 2 \( A \)-module. Let \( G \) be the Weil restriction of \( G_m \) from \( B \) to \( A \). Then the quotient \( G_{(B)} = G/G_m \) exists and is a smooth affine group scheme over \( A \). If \( B \) has basis 1, \( h \) with \( h^2 = mh - n \), then \( G_{(B)} = \text{Spec} \, A[v, w] \) where

\[
v = w^2 + mvw + nw^2 \quad \text{and} \quad w = l + l \frac{v}{1 + mw + mw^2}.
\]

**Proof:** The result is Zariski-local on \( \text{Spec} \, A \), so by localizing we may assume that \( B \) is free with a basis 1, \( h \). An element \( x + yh \) is invertible iff its norm \( z = x^2 + mxy + ny^2 \) is invertible, and thus \( A[G] \) is \( A[x, y, z]/(z(x^2 + mxy + ny^2) - 1) \). The functions \( v = y^2z \) and \( w = xyz \) are invariant.
under the subgroup \( G_m \) defined by \( y = 0 \), and it is a simple computation to show that \( u \) and \( w \) satisfy the relation and have \( \Delta u, \Delta w \) as stated. (Note for this that \( z \) is the reciprocal of the norm and hence is a character, i.e., \( \Delta z = z \otimes z \).) We have also \( 0 = \varepsilon(x) = \varepsilon(w) \), and the coinverse sends \( v \) to \( v \) and \( w \) to \( -w - mv \).

Suppose first \( A = K \) is a field. Then \( K[v, w] \) is a proper nontrivial Hopf subalgebra of \( K[G] \). Hence \( \text{Spec} \ K[v, w] \) is a group scheme; being a proper nontrivial quotient of \( G \), it is smooth and connected of dimension one. The polynomial \( w^2 + mvw + nv^2 - v \) is easily seen to be irreducible over \( K \), so it is the defining relation for the ring.

Now for general \( A \) let \( G' \) be \( \text{Spec} \ A[V, W]/W^2 + mVW + nV^2 - V \), with \( A \) defined by the formulas for \( v \) and \( w \). The identities needed for this to be a group scheme are satisfied, since for indeterminate \( m \) and \( n \) they hold over \( K = \mathbb{Q}(m, n) \) and hence over \( \mathbb{Z}[m, n] \). The ring is flat over \( A \), being actually free of rank 2 over \( A[V] \). By the argument over fields, all fibers are connected of dimension one, and \( G' \) is smooth over \( A \) since \( W \) spans the augmentation ideal modulo its square.

Sending \( V, W \) to \( v, w \) gives a homomorphism \( G \to G' \). Clearly \( G_m \) is in the kernel. But if evaluation at some point gives \( 0 = v = w \), then \( 0 = y^2 = xy \), since \( z \) is invertible. The \( x, y, z \) relation then gives \( 1 = zx^2 \), so \( x \) is invertible, and from \( xy = 0 \) we get \( y = 0 \). Thus the kernel is precisely \( G_m \). Both \( G \) and \( G' \) are smooth; one observes the map on Lie algebras is surjective, and by translation the same is true on all fibers, so \( G \to G' \) is smooth [4, p. 114] and hence flat. Since the fibers of \( G' \) are connected, the map is faithfully flat. Thus \( G' \) is the \((fppf)\) quotient \( G/G_m = G_{(B)} \).

Structure of the Fibers

The passage from \( B \) to \( G_{(B)} \) obviously commutes with base change in \( A \), and in particular the fibers of \( G_{(B)} \) are described by the following result.

**Proposition 3.2.** Let \( A = K \) be a field. Then \( G_{(B)} \) is

(a) \( G_m \) if \( B = K \times K \),
(b) a twisted torus if \( B \) is a separable quadratic field extension,
(c) \( G_a \) if \( B \) has nontrivial nilpotents, and
(d) a twisted additive group if \( B \) is an inseparable field extension (so \( \text{char } K = 2 \)).

**Proof.** If \( B \) is \( K \times K = K[h]/h^2 - h \), we have \( m = 1 \) and \( n = 0 \); thus by inspection \( G_{(B)} = G_m \). Even more trivially, if \( B \) is \( K[h]/h^2 \), we have \( m = n = 0 \) and \( G_{(B)} = G_a \). We now need a purely computational result:
Lemma 3.3. Suppose the basis in \( B \) is changed from \( h \) to \( h' = \alpha h + \beta \), with \( \alpha \) invertible. Then

\[
\begin{align*}
m' &= \alpha m + 2\beta, \\
n' &= \alpha^2 n + \alpha \beta m + \beta^2, \\
v &= \alpha^2 v', \\
w &= \alpha w' + \alpha \beta v'.
\end{align*}
\]

Now suppose \( B \) is a separable quadratic extension. After base extension to \( B \), the algebra becomes \( B \otimes_k B \simeq B \times B \), so \( G_{(h')} \) becomes \( G_m \); hence certainly \( G_{(h')} \) is a torus over \( K \). If over \( K \) we have \( G \simeq G_m \simeq G_{(h')} \) with \( B' = K[h'/((h')^2 - h')] \), consider the functions \( v', w' \) in \( K[G] \) corresponding to this. Over the larger field the algebras are isomorphic, and we have a linear change of variable from \( h \) to \( h' \). Since it takes \( v, w \) to \( v', w' \), and all these are defined over \( K \), we see that \( \alpha \) and \( \beta \) lie in \( K \); but that is impossible, since \( B \) is not isomorphic to the split algebra over \( K \). Thus \( G_{(h')} \) is anisotropic. The same argument shows that \( G_{(h')} \) is a twisted form of \( G_a \) if \( B \) is an inseparable field extension.

The argument here actually proves the following result:

Porism 3.4. Suppose \( G_{(h')} \simeq G_{(h'')} \) over a field. Then \( B \simeq B' \), and \( v, w \) are related to \( v', w' \) as in Lemma 3.3.

Classification Theorem

Theorem 3.5. Let \( A \) be an integrally closed domain with fraction field \( K \). Let \( G \) be a smooth finitely presented group scheme over \( A \) with connected one-dimensional fibers. Assume that \( G_K \) is a rational curve. Then \( G \) is of the form \( G_{(B)} \) for a unique algebra \( B \).

Proof. Note first that \( G_k \) is affine, since the projective line cannot carry a group scheme structure. Also, since \( G \) is finitely presented, it comes by base extension from a group scheme over a subring finitely generated over \( \mathbb{Z} \); the subring remains finitely generated if we replace it by its integral closure in any finite field extension \([5, \S8 and \S7]\). Thus we may assume \( A \) is noetherian.

The sections of the structure sheaf of \( G \) on the first two infinitesimal neighborhoods of the identity give us \( A \)-modules which (whether or not \( G \) is affine) we denote by \( I/I^2 \) and \( I/I^3 \). By smoothness these are projective. We let \( B \) be the rank 2 \( A \)-module \( (I/I^2) \otimes_A (I/I^3)^\vee \), where the superscript \( \vee \) denotes the dual module. Clearly formation of this \( B \) commutes with localization of \( A \).
Over the field $K$ we can by Corollary 2.9 write $G_K$ in the desired form, say with coordinates $v, w$ and constants $m, n$. If $f$ and $g$ are dual to the classes of $w$ and $v$ in $I_K/I^2_K$, then $[w] \otimes f$ and $[w] \otimes g$ are a basis of $B_K$. We define a multiplication of $B_K$ by letting $[w] \otimes f$ be the unit and making $h = [w] \otimes g$ satisfy $h^2 = mh - n$. Porism 3.4 shows easily that this multiplication is canonical, i.e., independent of the particular $v$ and $w$ chosen.

Now let $q$ be a height one prime of $A$. Corollary 2.9 shows that our group has the desired form over the discrete valuation ring $A_q$. If we choose $v$ and $w$ as in Corollary 2.9, the $m$ and $n$ are in $A_q$, and thus $B_q$ is an $A_q$-subalgebra of $B_K$. Now $A$ is integrally closed noetherian, and $B$ projective, so we have $B = \bigcap B_q$ inside $B_K$. Hence $B$ is an $A$-subalgebra of $B_K$. At this point we see that if we had started with $G = G_{(B')}$, we would here recover $B'$ as our $B$; thus the uniqueness is demonstrated.

In any case we can now form the group $(b)$ for our algebra $B$. Suppose first $G_K$ is a torus. Then there is an isomorphism $G_K \simeq (G(b))_K$, unique up to inversion, and by construction it extends to isomorphisms over $A_q$ for all height one primes. Since both $G$ and $G(b)$ are smooth with connected fibers, it follows by a theorem of Raynaud [8, p. 130] that in fact the map extends to an isomorphism over $A$. In the additive case, the $K$-isomorphism is unique only up to scalars. But the construction makes the $A$-modules $I/I^2$ for the two groups isomorphic, and we take a map which matches them up; the same argument then remains valid.

**Corollary 3.6.** All such groups $G$ are affine.

This is special to the one-dimensional case. Indeed, Raynaud [8, p. 116] has given an example where $A = \mathbb{C}[X_1, X_2]$ and $G$ is a smooth non-affine model of $\mathbb{G}_m \times \mathbb{G}_m$ with connected fibers.

**Corollary 3.7.** Let $A$ be an integrally closed domain with perfect fraction field. Then the smooth affine group schemes over $A$ with connected one-dimensional fibers are in one-to-one correspondence with the rank 2 $A$-algebras.

**Remarks.** (1) It is a pure formality to extend the theorem and its corollaries to non-affine integral normal base schemes.

(2) One can check that when $A$ is a discrete valuation ring, the operation of blowing up the identity in $G$ corresponds to replacing $B$ by $A + \pi B$. In general, if $G(b)$ and $G(b')$ are two models of $G_K$, the identification of generic fibers extends to a map $G(b) \to G(b')$ iff $B \subseteq B'$.

(3) It is well known [6, X.7.2] that one-dimensional tori over $\text{Spec } A$ are classified by rank 2 etale coverings. The theorem shows that more general smooth models are classified by the ramified coverings
Spec $B \to \text{Spec } A$. It would be interesting to find a more geometric process for recovering $B$ from the model.

**Models of $G_a$ and $G_m$**

**Corollary 3.8.** Let $A$ be an integrally closed domain. Smooth models of $G_a$ with connected fibers are classified by $\text{Pic}(A)$, the group of invertible fractional ideals modulo scalars. In particular, they are locally isomorphic to $G_a$ in the Zariski topology.

*Proof:* A subalgebra $B$ of $K[h]/h^2$ is spanned by 1 and the multiples of $h$ that it contains. These form an invertible fractional ideal, and an algebra isomorphism will change the ideal by a scalar multiplication. Explicitly, we obtain that $A[G]$ is the symmetric algebra on $I/I^2$.

Under more special hypotheses, this result was proved in [10].

**Corollary 3.9.** Let $A$ be an integrally closed domain. Smooth models of $G_m$ with connected fibers correspond to invertible ideals in $A$.

*Proof:* Here a subalgebra $B$ of $K[h]/h^2 - h$ must have the form $A \oplus Jh$ for some invertible ideal $J$ contained in $A$. If $t$ is the basic character of $G_m$, we find explicitly $A[G] = A[t^{-1}, J^{-1}(t - 1)]$.

*Remark.* The $G$ in this corollary is a group scheme version of a congruence subgroup: $G(A) = \{a \in G_m(A) \mid a \equiv 1 \pmod{J}\}$. Locally (and globally if $\text{Pic}(A) = 1$) we have $J$ principal, and $A[G]$ has the form $A[x, y]/(x(by + 1) - 1)$.

**Lifting to Characteristic Zero**

**Corollary 3.10.** Let $k$ be a field of positive characteristic. Let $G_k$ be a smooth connected affine group over $k$ of dimension one. The following are equivalent:

1. $G_k$ lifts to a smooth affine group over any valuation ring of mixed characteristic with residue field $k$.
2. $G_k \cong G_{(B)}$ for some $k$-algebra $B$.
3. $G_k$ is a rational curve.

*Proof:* We know by Proposition 3.2 and Corollary 2.9 that (2) and (3) are equivalent. One can lift $G_{(B)}$ just by lifting the $m$ and $n$ in the quadratic polynomial defining $B$, so (2) implies (1). Suppose conversely that $G$ lifts $G_k$. Replacing $G$ by its connected component [6, VI, 3.10], we may assume
$G_k$ is connected. Then the theorem shows $G$ has the form $G_{(B)}$, and the same is true of its reduction.

Observe that a group $G_k \simeq G_{(R)}$ actually lifts to any ring that maps onto $k$.

4. ARBITRARY MODELS OF $G_m$ OVER VALUATION RINGS

We now return to the setting and notation of the first two sections and use the blow-up method to analyze all models of $G_m$ over the discrete valuation ring $A$. The same procedure can be applied to models of $G_d$, but they have already been studied in [10] by a different method.

Reduced Polynomials

We first need an algebraic result for which we can let $A$ be any integral domain with fraction field $K$. Let $u_0, u_1, \ldots, u_m$ be indeterminates, and let $r(0), r(1), \ldots, r(m-1)$ be integers greater than 1. Call a monomial reduced if each $u_i$ for $i < m$ occurs in it with exponent less than $r(i)$, and call a polynomial reduced if it involves only reduced monomials. Let $B$ be the quotient of $A[u_0, \ldots, u_m]$ by relations of the form

\[
a_1 u_1 = u_0^{r(0)} + P_0(u_0),
\]

\[
a_2 u_2 = u_1^{r(1)} + P_1(u_0, u_1),
\]

\[\vdots\]

\[
a_m u_m = u_m^{r(m-1)} + P_{m-1}(u_0, u_1, \ldots, u_{m-1}),
\]

where the $a_i$ are nonzero elements of $A$ and the $P_i$ are reduced polynomials.

Assign to $u_{i+1}$ the weight $r(0) r(1) \cdots r(i)$, so that any reduced polynomial in $u_0, \ldots, u_i$ has weight less than that of $u_{i+1}$. By induction one shows now that every polynomial has the same image in $B$ as a reduced polynomial. Indeed, it is enough to prove this for monomials. If \[\prod u_i^{s(i)}\] has say $s(j) \geq r(j)$, replace it by

\[
\prod_{i \times j} u_i^{s(i)} \cdot [u_j^{s(j) - r(j)} [a_{j+1} u_{j+1} - P_j(u_0, \ldots, u_j)].
\]

All terms but the first have smaller weight, and the first term has the same weight and smaller total degree.

In the map to $B \otimes_A K \simeq K[u_0]$, the image of $u_i$ has by induction $u_0$-degree equal to the weight of $u_i$. Hence the reduced monomials all have different degrees and are independent. Thus $B$ is a free $A$-module with basis the reduced monomials. Its localization $B[u_0^{-1}]$ then is flat and injects into $B[u_0^{-1}] \otimes_A K = K[u_0, u_0^{-1}]$. We restate this result in the way it will be used:
Proposition 4.1. Suppose an $A$-subalgebra of $K[u_0, u_0^{-1}]$ is generated by $u_0, u_0^{-1}$, and elements $u_1, \ldots, u_m$ satisfying equations (*). Then these generators and relations define the algebra.

Actually the reduced polynomials are a basis of $B$ even when some $a_i$ are zero, or indeed for any $a_i$ in any ring $A$; this follows by base change from the result over $\mathbb{Z}$ with indeterminates adjoined. In essence this result is contained in [10].

Residue Characteristic Zero

We turn now to a model $G$ of $G_m$ over the valuation ring $A$. By Proposition 2.3 we know that $G$ is represented by a subalgebra $A[G]$ of $K[G_m] = K[t, t^{-1}]$ containing $t$ and $t^{-1}$. The structure of $G$ is particularly simple when char($k$) = 0, and we treat that case first.

Theorem 4.2. Suppose char($k$) = 0. Let $G$ be a model of $G_m$ over $A$, and assume $G \neq G_m$. Then $A[G] = A[t^{-1}, t, y]/(x^d y - t^n + 1)$ for some $d, n \geq 1$.

Proof: By the general theory we know $G$ is reached from $G_m$ by a standard blow-up sequence, and we merely have to verify inductively that each stage has this form. The only proper $k$-subgroups of $G_m$ are the $\mu_n$ for various $n \geq 1$, defined by $t^n - 1 = 0$. Thus the first blow-up is $A[t^{-1}, t, z]$ with $\pi z = t^n - 1$. One has $\Delta z_1 = z_1 \otimes 1 + 1 \otimes z_1 + \pi z_1 \otimes z_1$, and thus over $k$ the first blow-up group reduces to $\mu_n \times G_k$. The image $H$ of $G_k$ in this reduction must map onto the image $\mu_n$ at the previous stage, and therefore $H = \mu_n$ or $H = \mu_n \times G_k$, since $G_k$ has no proper subgroups in characteristic zero. If we do not yet have $G$ at this stage, the subgroup is proper and hence equals $\mu_n$, defined by $z_1 = 0$. Thus we adjoin $z_2 = \pi^{-1} z_1$, getting $A[t^{-1}, t, z]/(\pi^2 z_2 - t^n + 1)$. Again this reduces to $\mu_n \times G_k$, and the same argument applies here and at every subsequent stage. □

The "Néron model" of [7] does not appear here because it is not of finite type.

Residue Characteristic $p$: The Statement

The analysis when char($k$) = $p$ is similar in outline but more complicated because $G_k$ has non-trivial subgroups. We first state the theorem, give then a lemma and the induction step, and conclude with further comments and corollaries.

Theorem 4.3. Suppose char($k$) = $p$. Let $G$ be a model of $G_m$ over $A$, ...
and assume \( G \neq \mathbb{G}_m \). Then \( A[G] \) can be written as \( A[t^{-1}, t, y_1, \ldots, y_m] \) with relations of the form

\[
\begin{align*}
\pi^{d(1)} y_1 &= t^n - 1, \\
\pi^{d(2)} y_2 &= y_1^{r(1)} + P_1(y_1), \\
&\vdots \\
\pi^{d(m)} y_m &= y_{m-1}^{r(m-1)} + P_{m-1}(y_1, \ldots, y_{m-1})
\end{align*}
\]

for which the following statement are true:

1. The \( d(i) \) and \( n \) are \( \geq 1 \), the \( r(i) \) are non-trivial powers of \( p \), and the \( P_i \) are polynomials without constant term which are reduced with respect to the \( r(j) \).

2. \( P_i \) is congruent modulo \( \pi \) to a \( p \)-polynomial in \( y_i \) plus a reduced polynomial in \( y_1, \ldots, y_{i-1} \).

3. \( Ay_i \) is congruent modulo \( \pi \) to \( y_i \otimes 1 + 1 \otimes y_i \) plus a reduced expression in \( y_1 \otimes 1, 1 \otimes y_2, \ldots, y_{i-1} \otimes 1, 1 \otimes y_{i-1} \) with no constant term.

4. The schemes \( \text{Spec} A[t^{-1}, y_1, \ldots, y_1] \) are group schemes occurring as some of the stages in the standard blow-up sequence from \( \mathbb{G}_m \to G \).

5. The special fiber \( G_k \) is \( \mu_n \times U \) for some unipotent \( U \) that contains a subgroup \( \mathbb{G}_a \) with \( U/\mathbb{G}_a \) finite.

Remarks. (1) By Proposition 4.1, if the \( y_i \) satisfy these relations, they are defining relations; in particular, the special fiber is found by reading the equations modulo \( \pi \). Observe also that \( G \) is a complete intersection.

(2) Statement (5) is included only for clarity; it is a consequence of the more detailed statements preceding it.

(3) We may assume that \( P_i \) includes no terms divisible by \( \pi^{d(i+1)} \).

Homomorphisms to \( \mathbb{G}_a \)

Lemma 4.4. Let \( G \) be as in the conclusion of Theorem 4.3. Let \( \varphi: G_k \to \mathbb{G}_a \) be a homomorphism. Then \( \varphi \) has the form \( \sum c_s y_m^{p_s} + h(y_1, \ldots, y_{m-1}) \) for some reduced polynomial \( h \) with no constant term.

Proof: The homomorphism \( \varphi \) is of course identified with an element in \( k[G] = k[t, y_1, \ldots, y_m] \). Since \( \varphi \) must be trivial on \( \mu_n \), it is constant on \( \mu_n \)-cosets; hence \( t \) does not occur when we write \( \varphi \) in its unique reduced form. Let \( \psi \) be the restriction of \( \varphi \) to the subgroup \( \mathbb{G}_a \) of \( U \) given by \( y_1 = \cdots = y_{m-1} = 0 \). Since \( \psi \) is additive, it is given by some \( p \)-polynomial \( \sum c_s y_m^{p_s} \). Let \( \alpha: U \to \mathbb{G}_a \) be the map which replaces \( y_i \)-coordinates by zero for \( i < m \); by condition (3) in the theorem this is a \( \mathbb{G}_a \)-space map, i.e., \( \alpha(u + g) = \alpha(u) + g \).
for \( u \) in \( U \) and \( g \) in \( G_a \). Hence \( \phi - \psi \alpha \) is constant on \( G_a \)-cosets and passes to \( U/G_a \), so it is given by a function \( h(y_1, \ldots, y_{m-1}) \). There is no constant term because \( \phi \) vanishes on the identity element where all \( y_i \) are zero.

The Induction

An arbitrary model \( G' \) can be built up by its standard blow-up sequence from \( G_m \). The first blow-up, together possibly with others immediately following, has the same form as when \( \text{char}(k) = 0 \) and gives \( y_1 \) in the theorem. Inductively we suppose we have some \( G \) satisfying the theorem as one stage in the blow-up sequence; if \( H \) is the image of \( G_k' \) in \( G_k \), we must show \( G/H \) satisfies the theorem. We may of course assume \( H \neq G_k \).

First we show that \( H \) is the kernel of a homomorphism \( \phi : G_k \to G_a \). For this, let \( G'' \) be the stage in the blow-up sequence with \( A[G''] = A[t^{-1}, y_1, \ldots, y_{m-1}] \). The kernel \( V \) of \( G_k \to G'' \) is a copy of \( G_a \) with coordinate \( y_m \). The subgroup \( H \) must map onto the image of \( G_k' \) in \( G_k'' \). But by the construction of blow-ups, all of \( G_k \) maps into this image. Thus \( V/H \) is all of \( G_k \), and \( G_k/H \cong V/H \cap V \). This quotient of \( V \cong G_a \) is non-zero, since \( H \neq G_k \), and so it is isomorphic to \( G_a \).

The homomorphism \( \phi \) has the form established in the previous lemma. It must involve \( y_m \), since \( V/H \cap V \) is non-trivial. Suppose first it is linear in \( y_m \), so \( H \) is defined by an equation \( 0 = \phi = y_m + h(y_1, \ldots, y_{m-1}) \). The blow-up then is given by adjoining \( y'_m = \pi^{-1}[y_m - h'(y_1, \ldots, y_{m-1})] \), where \( h' \) is a lifting of \( h \). We can then drop \( y_m \) as a generator, replacing it by \( y'_m \). The equation involving \( y_m \) is replaced by

\[
\pi^{d(m)+1}y_m = y'_{m-1} + P_m(y_1, \ldots, y_{m-1}) + \pi^{d(m)}h'(y_1, \ldots, y_{m-1}).
\]

The statements in the theorem all trivially remain true.

Now suppose that \( \phi \) is of higher degree in \( y_m \). Then the blow-up has a new generator satisfying

\[
\pi y_{m+1} = y'^{r(m)} + P_m(y_1, \ldots, y_m),
\]

where the right-hand side is a lifting of \( \phi \) (which we may assume is again reduced with no constant term). The statements in the theorem follow from the lemma; in particular, (3) follows from \( \Delta \phi = \phi \otimes 1 + 1 \otimes \phi \). Thus Theorem 4.3 is proved.

Corollaries and Remarks

**Corollary 4.5.** For \( G \) as in the theorem, there is an exact sequence

\[
1 \to \mu_n \to G \to G' \to 1
\]

where \( G' \) is a model of \( G_m \) with unipotent special fiber.
Proof. Take $G'$, with variables $t'$ and $y'_i$, to have the same equations as $G$ except that $n$ is set $= 1$. We map $G \to G'$ by $t' = t^n$ and $y'_i = y_i$. Looking at reduced polynomials, we see that $A[t, y_1, \ldots, y_m]$ is free over $A[t', y'_1, \ldots, y'_m]$ with basis $1, t, \ldots, t^{n-1}$; the same then remains true after adjoining $(t')^{-1}$. In particular $G \to G'$ is faithfully flat. The kernel is obviously $\mu_n$; explicitly, the map to $A[X]/X^n - 1$ sends $t$ to $X$ and $t^{-1}$ to $X^{n-1}$ and $y_i$ to 0. 

**Corollary 4.6.**

(1) The special fiber is smooth iff $n$ is prime to $p$ and each $P_i$ contains a term $b_i y_i$ with $b_i$ invertible.

(2) The special fiber is connected iff $n$ is a power of $p$ and the $y_i$-terms in $P_i$ all have coefficients divisible by $n$.

**Proof.**

The condition on $n$ is of course the obvious restriction on $\mu_n$; the rest refers to $U$. But $U$ has a subgroup $G_a$ with $U/G_a$ finite, so $U$ is smooth or connected iff $U/G_a$ is. The finite group scheme $U/G_a$ has a filtration with successive quotients represented by $k[y_i]/(y_i^{r(i)} + y_i$-terms in $P_i$ (mod $\pi$)), so it is smooth or connected iff each of these is so. Thus the condition reduces to the standard one for subgroups of $G_a$.

The two requirements are incompatible if $m > 1$ or $n > 1$, so we recover the classification Theorem 2.5 of smooth models with connected fibers.

**Remarks.** (1) Induction shows that the number of steps in the standard blow-up sequence from $G_m$ to $G$ is precisely $d(1) + d(2) + \cdots + d(m)$. If $d(m) = 1$, the stage preceding $G$ is that where the generator $y_m$ and its relation are omitted. If $d(m) > 1$, the preceding stage has $d(m)$ decreased by 1 and terms divisible by $\pi^{d(m) - 1}$ dropped from $P_{m-1}$.

(2) As we observed at the start of the section, the polynomial $y_i$ in $K[t]$ has degree $nr(1) \cdots r(i-1)$, larger than the degree of any reduced polynomial in $t$ and the lower $y_j$. Hence when we write the elements of $K[t] \cap A[G] = A[t, y_1, \ldots, y_m]$ in reduced form, those of degree $< n$ involve only $t$, those of degree $< nr(1)$ involve only $t$ and $y_1$, and so on. Thus the generators of $K[t] \cap A[G]$ are built up in the obvious algebraic way: go up by degrees, taking in new elements if they are not in the ring generated by those previously available. Indeed, this starts with $t$, then gives nothing else until $y_1$ in degree $n$, then nothing until $y_2$ in degree $nr(1)$, and so on. In particular, this shows that the numbers $n, r(i), d(i)$ are unique.

Also we observe that $y_i$ as a polynomial in $K[t]$ has non-zero constant term. For otherwise we would have $t^{-1}y_i$ in $K[t] \cap A[G]$; being of lower degree, it should be in $A[t, y_1, \ldots, y_{i-1}]$, which is impossible since its reduced expression in $K[t, y_1, \ldots, y_{i-1}]$ involves $\pi^{-d(i)}$.

(3) The polynomials $P_i$ satisfy further conditions, but they seem to be too complicated to be worth working out. We should at least mention that
two statements known [10] for models of $G_a$ are not true here: the $P_i$ need not be $p$-polynomials in $y_i$, and the $d(i)$ for $i > 1$ need not be less than $\text{ord}_A(p)$. For example, take $A = \mathbb{Z}_3^{[3^{1/2}]}$, with $\pi = 3^{1/2}$. Let $G = \text{Spec } A[t, y_1, y_2]$, with $\pi y_1 = t - 1$ and $\pi^d y_2 = y_1^d + \pi y_2 + y_1$; for any $d$ this is a model of $G_m$ (in fact $y_2 = y_1 + 1 + y_2 + \pi y_2$). What happens here is that the equation $y_1^d + \pi y_2 + y_1 = 0$ actually defines a subgroup which is flat (a model of $\mu_3$) inside $\text{Spec } A[t, y_1]$.

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