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On the Geometry of Unitary Groups*

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The present paper grew out an attempt to understand the techniques used by O'Meara's school for the isomorphism problem (for example, [5], [8], [9], [12]). Nevertheless, the final result obtained is new in that it describes not only isomorphisms between full subgroups of unitary groups, but also monomorphisms among them. The result in this generality is needed to study monomorphisms (and even isomorphisms) between full subgroups of algebraic groups of type G_2; this is done in a subsequent paper [14].

As far as the understanding of O'Meara's method is concerned, we observe that behind this method there is an algebraic geometry on the set of some subgroups of unitary groups. This geometry is reminiscent of Tits' geometries but differs from them. We construct this geometry in Section 2 and exhibit in detail its different aspects.

The study of the unitary group as an algebraic group requires that we should extend the definition of unitary groups also to the split case. This is done in Section 1. The main result of this section, Theorem 1.2.4, extends the fundamental theorem of projective geometry to projective spaces over semilocal rings.

In Section 3 we define "full" subgroups of unitary groups. They are full in the sense that the algebraic geometry on the set of subgroups described in Section 2 can be reconstructed from the corresponding subgroups of a full group using only group theoretic operations. We actually go slightly further and describe the abstract properties a group with a distinguished set of subgroups must satisfy to be a full subgroup of a unitary group. This axiomatization is neither effective nor pleasant-looking, but we believe it worth pointing out. Actually, some kind of characterization (usually, implicit) is common to most techniques used in the homomorphism problem. Finally, in the same Section 4 we prove our main result which describes monomorphisms between unitary groups. The result is stated in two forms. One, in the spirit of A. Borel and J. Tits [4] and another in a more classical form. It is this last form which is used to handle the groups of type G_2.

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Notations and Conventions

Let A be a ring. If X is a scheme over A we express this fact by writting X_A. For $B \supseteq A$ we denote by $X_A(B)$ or $X(B)$ the set of points of X in B. If $\varphi\colon A \to B$ is a ring homomorphism we denote by $\varphi(a)$ or by a^φ the image of a under φ, and by ${}^\varphi X$ a scheme over B obtained by base change φ; then $\varphi^\circ\colon X(A) \to {}^\varphi X(B)$ is the corresponding map of points. If $B \subsetneq A$ and A is of finite rank over B we denote by $R_{A/B}X$ the Weil restriction from A to B (that is, we consider X_A as a scheme over B via $R_{A/B}$).

Since it would otherwise complicate notations and statements we usually deal with the points of X in A (although everything is functorial in A). For example, we talk about a stabilizer of a point in a vector space $V = V_k$ and for verification we take a point $x \in V(k)$, instead of taking a point $x \in V(\tilde{k})$ with $\tilde{k} \supseteq k$.

By an algebraic group we understand a reduced affine group scheme of finite type. The multiplicative group is denoted $\mathbb{G}_m$ and the additive group $\mathbb{G}_a$.

For an algebraic (resp. abstract) group G and an algebraic (resp. abstract) subset H of G we denote by $Z_G(H)$, $N_G(H)$, $D^i(G)$ and $C(G)$ the algebraic (resp. abstract) subgroup of G which is respectively the centralizer of H in G, the normalizer of H in G, the i-th derived group of G and the center of G. If H_1 and H_2 are algebraic (resp. abstract) subgroups of G, then $[H_1, H_2]$ denotes their commutator.

For a free A-module M of finite rank, $\mathbb{P}(M)$ denotes its projective space, namely, the set of free rank one direct summands of M. If $rk_A M = n$, the associated scheme over A is denoted by $\mathbb{P}_A^{n-1}$ so that $\mathbb{P}(M) = \mathbb{P}_A^{n-1}(A)$.

$|X|$ denotes the cardinality of X.

1. Unitary Groups

The main point of this section is to extend the theory of unitary groups to all separable algebras of dimension two over a base field k, that is, in addition to the usual unitary groups which correspond to quadratic separable field extensions of k we consider unitary groups over $k \oplus k$.

1.1. *Hermitian Spaces*

Let k be a field of any characteristic and K a quadratic separable algebra with identity over k. Denote by σ the nontrivial k-automorphism of K; for $a \in K$ let $N(a) = aa^\sigma$ and $\mathrm{Tr}(a) = a + a^\sigma$. If $K = k \oplus k$, fix $u \in K$ such that $u^2 = u$, $u \neq 1$. Then $u^\sigma = 1 - u$.

1.1.1. Let V be a free K-module of rank n. A submodule V_1 of V is called a *subspace* of V if it is a free submodule of V; a one-dimensional subspace is called a *line* and a two-dimensional subspace a *plane.*

LEMMA. *If V_1 is a subspace of V then there exists a subspace V_2 of V such that* $V = V_1 \oplus V_2$.

Proof. If K is a field the assertion is clear. If $K = k \oplus k$, consider V as a vector space over k endowed with a linear operator $S : v \to uv$ for $v \in V$. The operator S is completely reducible, it has $n = \frac{1}{2} \dim_k V$ eigenvalues 1 and n eigenvalues 0. It has the same number $m = \frac{1}{2} \dim_k V_1$ eigenvalues 1 and 0 on V_1. Let V_2 be an S-invariant k-complement to V_1 in V. Then since $SV_2 \subseteq V_2$ we see that V_2 is a K-module and since the number of eigenvalues 1 and 0 of S on V_2 is the same, it follows that V_2 is a free K-module, as required.

1.1.2. We endow V with a non-degenerate Hermitian scalar product F: $V \times V \to K$, that is for $x, y, z \in V$ and $a, b \in K$

(i) $F(x, y) = F(y, x)^\sigma$

(ii) $F(ax + by, z) = aF(x, z) + bF(y, z)$

(iii) $F(L, V) = K$ for any line $L \subseteq V$.

Denote by f: $V \to k$ the corresponding Hermitian form; thus $f(x) = F(x, x)$.

For a subset W of V let $W^\perp = \{v \in V \mid F(v, W) = 0\}$ be its orthogonal complement. A subspace W of V is called *totally isotropic* if $F(W, W) = 0$, *regular* if F restricted to U is non-degenerate, and *anisotropic* if $f(L) \neq 0$ for every line L in W. Since $f(L) \subseteq k$, every line L of V is either isotropic (i.e., $F(L, L) = 0$) or regular ($=$ anisotropic).

1.1.3. LEMMA. *V has an orthogonal K-basis.*

Proof. Since F is non-degenerate and K is a separable extension of k, there exists an anisotropic line $L = Ke$ in V. Then $x - F(x, e)f(e)^{-1}e$ is in $L^\perp$ for all $x \in V$. Hence $V = L \perp L^\perp$ and the result follows by induction on the rank of V.

1.1.4. Call two orthogonal bases $\{e_i\}$, $\{e_i'\}$ *associated* if for all but two indices i there exists an index $j = j(i)$ such that $e_i = e_{j(i)}'$. In other words we change our basis only in a two-dimensional space generated by some two basis vectors.

PROPOSITION. *Except when $|k| = 2$ any two orthogonal bases of V can be connected by a chain of $m < \infty$ associated bases. Moreover, if V is anisotropic, or if the characteristic k is not two, a chain may be chosen with $m \leqslant \frac{1}{2} n(n-1)$ where* $n = \operatorname{rank}_K V$.

Proof. The argument in [11; 1.14, 1.15] can be easily modified to our situation. We use induction on n and may assume $n \geqslant 3$. Let $\{e_i\}$ and $\{e_i'\}$ be orthogonal bases and $e_1' = \sum a_j e_j$ where $a_j \in K$. We will construct a sequence of changes to associated bases enabling us to assume $e_1' = e_i$ for some i. Assume a_1, a_2, a_3 and $N(a_1)$ are all nonzero, for otherwise the result is clear. If V is anisotropic or the characteristic of k is not two, then at least one of $f(a_1e_1 + a_2e_2)$,

$f(a_2e_2 + a_3e_3)$, $f(a_1e_1 + a_3e_3)$ is nonzero. Assume $f(a_1e_1 + a_2e_2) \neq 0$ and set $\bar{e}_1 = a_1e_1 + a_2e_2$, $\bar{e}_2 = e_2 - F(e_2, \bar{e}_1)f(\bar{e}_1)^{-1}\bar{e}_1$, $\bar{e}_i = e_i$ for $i \geqslant 3$. Then the basis $\{\bar{e}_i\}$ is associated to $\{e_i\}$ and in the expression $e_1' = \sum \bar{a}_j\bar{e}_j$ fewer coefficients are nonzero. After at most $n - 1$ such steps we obtain a basis with e_1' as first component. Applying the same argument to $(Ke_1')^\perp$ we will get the required chain of associated bases in at most $\frac{1}{2}n(n-1)$ steps. Now assume V is isotropic and k has characteristic two. The above proof only fails when $f(a_1e_1) = f(a_2e_2) = f(a_3e_3)$. Since $|k| \geqslant 4$, we may change $Ke_1 \perp Ke_2$ to $Kf_1 \perp Kf_2$ with $a_1e_1 + a_2e_2 = b_1f_1 + b_2f_2$ and $f(a_1e_1) \neq f(b_1f_1)$. The argument then continues as before.

Examples. Let V have basis e_1, e_2, e_3 where $F(e_i, e_j) = \delta_{ij}$ and take $K = k(a)$ with $a^2 = a + 1 = a^\sigma$. Then $e_1' = e_1 + e_2 + e_3$, $e_2' = e_1 + ae_2 + a^\sigma e_3$, $e_3' = e_1 + a^\sigma e_2 + ae_3$ is also an orthogonal basis of V. If $k = \mathbb{F}_2$ it is impossible to construct a chain of associated bases from $\{e_i\}$ to $\{e_i'\}$. When $k = \mathbb{F}_2(X)$ the minimal length of a chain is 4.

In general, when $\dim V = 3$ and $|k| \neq 2$, it is clear from the proof of the Proposition that a chain of length at most four exists between any two orthogonal bases of V.

1.1.5. Lemma. *Let M be a subspace of V. Then $M^\perp$ and $M \cap M^\perp$ are also subspaces of V and $M^{\perp\perp} = M$.*

Proof. If $L \subseteq M$ is a regular line, then there is an orthogonal decomposition $V = L \perp V_1$ with $M = L \perp M_1$ and M_1 a subspace of V_1. By induction we therefore have $V = L_1 \perp \cdots \perp L_s \perp W$ and $M = L_1 \perp \cdots \perp L_s \perp N$ with the L_i anisotropic lines and N a totally isotropic subspace of W. Since W is nondegenerate, it has a decomposition $W = H_1 \perp \cdots \perp H_t \perp U$ where $H_i = Ku_i + Kv_i$, $F(u_i, v_i) = 1$ and $u_1, \ldots, u_t$ is a basis of N. Clearly, $M^\perp = N \perp U$ is now a subspace of V. Also, $M \cap M^\perp = N$ is a subspace and $M^{\perp\perp} = M$.

1.1.6. Proposition (i) *A plane P in V is either regular, degenerate but not totally isotropic, or totally isotropic.*

(ii) *If P is degenerate but not totally isotropic, then P is contained in a regular three-dimensional space and has a basis e_1, e_2 such that $f(e_1) \in k^*$, $f(e_2) = 0$ and $F(e_1, e_2) = 0$.*

(iii) *If P is totally isotropic then P is contained in a regular four-dimensional space with a basis e_1, e_2, e_3, e_4 such that $F(e_1, e_4) = F(e_2, e_3) = 1$ and $F(e_i, e_j) = 0$ otherwise, and $P = Ke_1 + Ke_2$.*

Proof. This is clear from the proof of 1.1.5.

Remark. In the case (ii) P contains exactly one isotropic line $P \cap P^\perp$ if K is a field. Otherwise P contains several isotropic lines, for example the lines generated by $aue_1 + e_2$ and $au^\sigma e_1 + e_2$ with $a \in k$.

1.1.7. LEMMA. *Let $rk_K V \geqslant 3$ and L_1, L_2 be lines in a plane P. Then $P = L_1 + L_2$ if and only if for all lines $L \subseteq L_1{}^{\perp} \cap L_2{}^{\perp}$ it follows that $L \subseteq P^{\perp}$.*

Proof. If $P = L_1 + L_2$ the assertion is clear. For the converse we need only consider $K = k \oplus k$.

Let $L_1 = Ke_1$ and assume first L_1 is regular. Then $V = L_1 \perp W$ and $P = L_1 \perp L_3$ where $L_3 = Ke_3 \subseteq W$. Let $L_2 = K(ae_1 + be_3)$ where $b \neq 0$. The result is clear if b is a unit; assume, therefore, $bb^{\sigma} = 0$. If $f(e_3) \neq 0$, then there exists $e_4 \in P^{\perp}$ with $f(e_4) \neq 0$; put $x = be_3 + e_4$. If $f(e_3) = 0$, then there exists isotropic $e_4 \in W$ with $F(e_3, e_4) = 1$; put $x = e_3 + be_4$. In both cases Kx is a line in $L_1{}^{\perp} \cap L_2{}^{\perp}$ but not in $P^{\perp}$.

We may now assume that both $L_1 = Ke_1$ and L_2 are isotropic. Thus $V = H \perp W$ with $H = Ke_1 + Ke_2$, $F(e_1, e_2) = 1$ and $f(e_2) = 0$. Let $L_2 = Kx$ where $x = ae_1 + be_2 + w$ with $w \in W$. We can assume $a \neq 0$, $bb^{\sigma} = 0$ and $F(w, W) \neq K$, for otherwise, $L_1 + L_2 = P$. Now $\mathrm{Tr}(ab^{\sigma}) = f(x) = 0$ and $ab^{\sigma} = 0$. Thus either $a \in kb$ and then $P = L_1 + L_2$, or $b = 0$ and $F(L_1, L_2) = 0$. In this last case $L_1 + L_2 \subseteq P^{\perp}$ and $P = Ke_1 + Ke_3$ with $e_3 \in W$ and $f(e_3) = 0$. Take $e_4 \in W$ with $F(e_3, e_4) = 1$. Then $e_1 + ue_4 \in L_1{}^{\perp} \cap L_2{}^{\perp}$ but $F(e_1 + ue_4, P) = Ku \neq 0$, contradicting our hypothesis.

1.2. *Projective Spaces*

Let V be a space over K of rank n. With it we associate the projective space $\mathbb{P}_K^{n-1}$. The points $\mathbb{P}_K^{n-1}(K)$ of $\mathbb{P}_K^{n-1}$ over K are free direct factors of V (cf. [6; *I*, Sect. 1.3.4]) that is, the lines of V. The set $\mathbb{P}_K^{n-1}(K)$ carries combinatorial geometry. It is defined (functorially) in the usual way: a (*linear*) *subspace* of $\mathbb{P}_K^{n-1}(K)$ is the set of lines of V contained in a fixed subspace of V. We frequently use the same symbol to denote both the subspace of V and the corresponding subspace of $\mathbb{P}_K^{n-1}(K)$. A subspace M of $\mathbb{P}_K^{n-1}(K)$ is called regular, isotropic, etc., if it comes from a subspace M of V with the same property.

1.2.1. LEMMA. *A non-degenerate Hermitian form F on V defines an autoduality, also denoted F, of $\mathbb{P}_K^{n-1}$. The autoduality F maps points of $\mathbb{P}_K^{n-1}$ to $(n-2)$-dimensional subspaces of $\mathbb{P}_K^{n-1}$ and conversely via $F(L) = L^{\perp}$, $F(M) = M^{\perp}$ where L is a line of V and M is a hyperplane of V.*

Proof. This is a reformulation of 1.1.5.

Remark. This autoduality permits us to distinguish isotropic and anisotropic points of $\mathbb{P}_K^{n-1}$. Isotropic points x are those which are mapped by F into a hyperplane $F(x)$ such that $x \in F(x)$.

1.2.2. We will need an extension of the Fundamental Theorem of Projective Geometry. The theorem established below is more general than we require here, but we have some further uses in mind. Let A be a commutative ring with identity and M a free A-module of finite rank n. A line in M is a free rank one direct summand of M and a plane is a free rank two direct summand. The projective space $\mathbb{P}(M)$ of M is the family of all lines in M. Let B be a second commutative ring with identity and N a free B-module. A mapping $\alpha\colon \mathbb{P}(M) \to \mathbb{P}(N)$ is called a *projectivity* if for all lines L, L_1 and L_2 in M with $L_1 + L_2$ a plane and $L \subset L_1 + L_2$ it follows that $\alpha L \subset \alpha L_1 + \alpha L_2$.

1.2.3. Two bases $\{e_i\}$ and $\{e_i'\}$ of M are called 2-associated if they differ, apart from order, only in at most two terms spanning the same subspace.

LEMMA. *Assume A is a semilocal ring. Then any two bases $\{e_i\}$ and $\{e_i'\}$ can be connected by a chain of 2-associated bases.*

Proof. By induction it suffices to show the basis $\{e_i\}$ can be connected by a chain of 2-associated bases to a basis containing e_1'. Let $e_1' = \sum a_i e_i$. If any coefficient a_i of e_1' is a unit, the result follows easily in a sequence of at most $n - 1$ steps. We now arrange a sequence of changes through 2-associated bases to a new basis $\{\bar{e}_i\}$, say, in which $e_1' = \sum \bar{a}_i \bar{e}_i$ and some $\bar{a}_i$ is a unit. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$ be all the maximal ideals in A. Note that $a \in A$ is a unit if and only if $a \notin \mathfrak{m}_i$, $1 \leqslant i \leqslant s$. Assume that in $e_1' = \sum a_i e_i$ the coefficient $a_1 \notin \mathfrak{m}_i$, $1 \leqslant i \leqslant t - 1$, but $a_1 \in \mathfrak{m}_t$. Then there exists a second coefficient a_2, say, with $a_2 \notin m_t$. By the Chinese Remainder Theorem there exists b, c in A with $b \equiv 1 \bmod \mathfrak{m}_i$ for $i \neq t$, $b \in \mathfrak{m}_t$, $c \equiv 1 \bmod \mathfrak{m}_t$ and $c \in \mathfrak{m}_i$ for $i \neq t$. Set $\bar{e}_1 = be_1 + ce_2$, $\bar{e}_2 = ce_1 + be_2$, $\bar{e}_i = e_i$ for $i \geqslant 3$. Then $\{\bar{e}_i\}$ is a basis 2-associated with $\{e_i\}$. If $e_1' = \sum \bar{a}_i \bar{e}_i$, then $\bar{a}_1 \notin \mathfrak{m}_i$ for $1 \leqslant i \leqslant t$. It is now clear how to complete the proof.

1.2.4. THEOREM. *Let A be a semilocal ring and $\alpha\colon \mathbb{P}(M) \to \mathbb{P}(N)$ a projectivity. Assume that* $\operatorname{rank}_A M = \operatorname{rank}_B N = n \geqslant 3$ *and that the family of lines of N in $\alpha\mathbb{P}(M)$ span N. Then there exists a unique ring homomorphism $\varphi\colon A \to B$ and a φ-semilinear mapping $\beta\colon M \to N$, which is unique up to multiplication by a unit in B, such that $\alpha Ax = B\beta(x)$ for any line Ax in M. Moreover, if α is injective, then φ and β are injective.*

Proof. There are a number of steps.

1.2.5. Let $\{e_i\}$ be a basis for M and $\alpha Ae_i = Bf_i$, $1 \leqslant i \leqslant n$. We first prove that $\{f_i\}$ is a basis for N. This is an immediated consequence of our hypotheses if we show for any line Ax in M that $\alpha Ax \subseteq \sum_i Bf_i$. By Lemma 1.2.3 there exists a chain of 2-associated bases of M starting with $\{e_i\}$ and ending with a basis containing x (x is part of a basis by Bass [1; p. 168]). At each step as we change bases, starting with $\{e_i\}$, it follows for the new basis $\{\bar{e}_i\}$, say, that $\alpha A\bar{e}_j \subseteq \sum_i Bf_i$, $1 \leqslant j \leqslant n$. This is a consequence of the definition of a projectivity and the fact that only two basis elements in a fixed plane are changed in each step. Hence, finally, $\alpha Ax \subseteq \sum Bf_i$.

1.2.6. We now construct a homomorphism $\varphi\colon A \to B$. Take $i \neq 1$. Then $A(e_1 + e_i)$ is a line and $\alpha A(e_1 + e_i) = B(bf_1 + cf_i)$. Since $Ae_1 \subset A(e_1 + e_i) + Ae_i$, it follows that $Bf_1 \subset B(bf_1 + cf_i) + Bf_i$. Hence b and similarly c are units. Replacing f_i by $bc^{-1}f_i$ we may assume $\alpha A(e_1 + e_i) = B(f_1 + f_i)$, $2 \leqslant i \leqslant n$. Similarly, for $a \in A$ we have $\alpha A(e_1 + ae_2) = B(f_1 + bf_2)$ for some unique $b \in B$. Define $\varphi\colon A \to B$ by $\varphi(a) = b$. Then $\varphi(0) = 0$ and $\varphi(1) = 1$. Similarly, for $i \geqslant 3$ define $\varphi_i : A \to B$ by $\alpha A(e_1 + ae_i) = B(f_1 + \varphi_i(a)f_i)$.

We now verify that φ is a homomorphism. Fix $i \geqslant 3$. Then $x = e_1 + a_2e_2 + a_ie_i$ is in the plane $A(e_1 + a_2e_2) + Ae_i$. Hence αAx is contained in $B(f_1 + \varphi(a_2)f_2) + Bf_i$, and similarly in $B(f_1 + \varphi_i(a_i)f_i) + Bf_2$. Hence $\alpha Ax = B(f_1 + \varphi(a_2)f_2 + \varphi_i(a_i)f_i)$. By a similar argument, $\alpha A(ae_2 + e_i) = B(\varphi(a) f_2 + f_i)$. Then, for a, b in A,

$$\begin{aligned} &B(f_1 + \varphi(a + b)f_2 + f_i) \\ &\quad = \alpha A(e_1 + (a + b)e_2 + e_i) \subset B(f_1 + \varphi(a)f_2) + B(\varphi(b)f_2 + f_i) \end{aligned}$$

and hence $\varphi(a + b) = \varphi(a) + \varphi(b)$. Likewise,

$$B(f_1 + \varphi(ab)f_2 + \varphi_i(b)f_i) = \alpha A(e_1 + abe_2 + be_i) \subset Bf_1 + B(\varphi(a)f_2 + f_i)$$

so that $\varphi(ab) = \varphi(a)\,\varphi_i(b)$. Setting $a = 1$ gives $\varphi = \varphi_i$ and hence φ is a ring homomorphism. Moreover, if α is injective and $\varphi(a) = 0$, then $\alpha Ae_1 = \alpha A(e_1 + ae_2)$ so that $a = 0$ and φ is also injective.

1.2.7. Let $x = \sum a_ie_i$ with Ax a line in M. We now show that

$$\alpha Ax = B\left(\sum_{i=1}^{n} \varphi(a_i)\, f_i\right).$$

If $a_1 = 1$ then by induction (since the cases $n = 2, 3$ are already known)

$$\alpha Ax \subset \alpha A\left(e_1 + \sum_{i=1}^{n-1} a_ie_i\right) + Bf_n = B\left(f_1 + \sum_{i=1}^{n-1} \varphi(a_i)\, f_i\right) + Bf_n\,.$$

Since, also, $\alpha Ax \subset B(f_1 + \varphi(a_n)f_n) + Bf_2 + \cdots + Bf_{n-1}$, the result follows. If a_1 is a unit, then $\varphi(a_1)$ is a unit and $\alpha Ax = \alpha Aa_1^{-1}x = B(\sum \varphi(a_i)f_i)$. Now assume some a_i is a unit with $i \neq 1$. Repeat earlier work in 1.2.6 with 1 and i interchanged. Then there exists a homomorphism $\psi\colon A \to B$ such that $\alpha A(ae_1 + e_i) = B(\psi(a)f_1 + f_i)$. Since, for $j \neq 1$, i,

$$\alpha A(e_1 + ae_j + e_i) = B(f_1 + \varphi(a)f_j + f_i) = B(f_1 + \psi(a)f_j + f_i)$$

we have $\varphi = \psi$ and the result follows as before.

Finally, assume no coefficient a_i is a unit. Consider a chain of 2-associated bases from $\{e_i\}$ to a basis containing x. In particular, consider a change from $\{e_i\}$ to $\{\bar{e}_i\}$ where $\bar{e}_1 = be_1 + ce_2$, $\bar{e}_2 = b'e_1 + c'e_2$, $\bar{e}_i = e_i$ for $i \geqslant 3$. Again there exists a homomorphism $\bar{\varphi}$: $A \to B$ such that

$$\alpha A(a\bar{e}_1 + e_3) = B(\bar{\varphi}(a)\bar{f}_1 + f_3) = B(\varphi(ab)f_1 + \varphi(ac)f_2 + f_3).$$

Hence $\bar{\varphi}(a)\bar{f}_1 = \varphi(ab)f_1 + \varphi(ac)f_2$. Setting $a = 1$ it follows that $\bar{f}_1 = \varphi(b)f_1 + \varphi(c)f_2$ and $\bar{\varphi} = \varphi$. Thus the validity of our formula for αAx is unaffected by changes through 2-associated bases and hence follows for all lines Ax in M.

1.2.8. Define β: $M \to N$ by $\beta(\sum a_ie_i) = \sum \varphi(a_i)f_i$. Then β is a φ-semilinear mapping and $\alpha Ax = B\beta(x)$ for all lines Ax in M. Assume $\tilde{\beta}$: $M \to N$ is also a $\tilde{\varphi}$-semilinear mapping with $\alpha Ax = B\tilde{\beta}(x)$ for all lines Ax in M. Then $\tilde{\beta}(e_i) = c_if_i$ with c_i a unit in B. Since also, for $i \neq j$, $\tilde{\beta}(e_i + e_j) = c_{ij}(f_i + f_j)$, it follows that $c_i = c$ and $\tilde{\beta} = c\beta$. Thus β is unique up to multiplication by a unit of B and then $\tilde{\varphi} = \varphi$.

Now assume α is injective. If Ax and Ay are lines with $\beta(x) = \beta(y)$, then $Ax = Ay$ and hence $x = by$ with b a unit in A. Then $(\varphi(b) - 1)\beta(y) = 0$ hence $\varphi(b) = 1$. Thus $b = 1$ and $x = y$. In the general case with $x, y \in M$ and $\beta(x) = \beta(y)$, take $z \in M$ such that $A(x + z)$ and $A(y + z)$ are lines. Then $\beta(x + z) = \beta(y + z)$ forces $x + z = y + z$. Hence $x = y$ and β is also injective.

1.2.9. We now examine β and φ in Theorem 1.2.4 when α is surjective. In general, φ will not be surjective. For example, take A a discrete valuation ring with quotient field B, M a free A-module and $N = M \otimes B$. The natural injection of M into N then induces a bijective projectivity from $\mathbb{P}(M)$ onto $\mathbb{P}(N)$. However, it does easily follow in general that every element of B is the quotient of two elements in $\varphi(A)$ if α is surjective.

Moreover, with a more restrictive choice of A (and possibly B), the surjectivity of α does force the surjectivity of φ (and then also β). In each of the following cases φ will be surjective (if α is):

(i) A is a finite direct sum of fields and α is bijective;

(ii) $A = B$ a local ring with maximal ideal J and A/J is an algebraic extension of its prime field;

(iii) $A = B$ a discrete valuation ring.

1.3. *Groups*

Let V be a K-space of rank n. We consider groups of linear transformations of V. All these groups are defined functorially (with respect to field extensions of k) and will be considered as group functors.

1.3.1. The groups $GL(V) \cong GL_K(n)$ and $SL(V) \cong SL_K(n)$ are connected affine algebraic K-groups. If we consider them as groups of transformations of the $2n$-dimensional k-space $R_{K/k}V$ then they become algebraic k-groups $R_{K/k}\, GL(V)$ and $R_{K/k}\, SL(V)$.

LEMMA. *If $K = k \oplus k$ then $R_{K/k}\, GL(V)$ and $R_{K/k}\, SL(V)$ are k-isomorphic to $GL_k(n) \times GL_k(n)$ and $SL_k(n) \times SL_k(n)$ respectively.*

Proof. Let $A, B \in GL_k(n)$. Then $uA + u^\sigma B \in GL_K(n)$ since $(uA + u^\sigma B)^{-1} = uA^{-1} + u^\sigma B^{-1}$ and the first result follows. In the second situation, take A, $B \in SL_k(n)$.

1.3.2. LEMMA. *$GL(V)$ acts transitively on the set of bases of V; $SL(V)$ acts transitively on the set of n-tuples of spanning lines of V.*

1.3.3. COROLLARY. *$\mathbb{P}_K^{n-1}$ is a homogeneous space for $GL(V)$ and $SL(V)$. In particular, $R_{K/k}\mathbb{P}_K^{n-1}$ is a form of $\mathbb{P}_k^{n-1} \times \mathbb{P}_k^{n-1}$, and if $K = k \oplus k$ then $R_{K/k}\mathbb{P}_K^{n-1} \simeq \mathbb{P}_k^{n-1} \times \mathbb{P}_k^{n-1}$.*

Proof. The first assertion follows since any line can be included in a set of n spanning lines by 1.1.1. The final assertion follows from 1.3.1 and then, consequently, $R_{K/k}\mathbb{P}_K^{n-1}$ is a form of $\mathbb{P}_k^{n-1} \times \mathbb{P}_k^{n-1}$.

1.3.4. Now let V be a Hermitian space with a Hermitian pairing F. The group $GU(V, F)$ consists of all $A \in GL(V)$ such that $F(Ax, Ay) = \lambda(A)\, F(x, y)$ for some $\lambda(A) \in K$. Automatically we have $\lambda(A) \in k^*$ (by 1.1.2(i), since F is nondegenerate) and that the map $\lambda\colon GU(V, F) \to k^*$ is a group homomorphism. We define $U(V, F) = \operatorname{Ker} \lambda$ and $SU(V, F) = \operatorname{Ker}\{\det\colon U(V, F) \to K^*\}$. The definitions of these groups are functorial.

LEMMA. *$GU(V, F)$ consists of those $A \in GL(V)$ such that $F(Ax, Ay) = 0$ whenever $F(x, y) = 0$. This description is functorial.*

Proof. Clearly $A \in GU(V, F)$ satisfies the condition. Suppose now that $A \in GL(V)$ satisfies the condition. Let $e_1, \ldots, e_n$ be an orthogonal basis of V (cf. 1.1.3). Then $F(Ae_i, Ae_j) = 0$ if $i \neq j$. Let $a_i = f(e_i)$. Then $e_j + a_j e_i$ and $e_i - a_i e_j$ are orthogonal so that $F(A(e_j + a_j e_i), A(e_i - a_i e_j)) = 0$. This implies $a_i^{-1} F(Ae_i, Ae_i) = \lambda(A)$ is independent of i. Thus $A \in GU(V, F)$.

1.3.5. LEMMA. *The groups GU, U and SU are algebraic k-subgroups of $R_{K/k}GL(V)$.*

Proof. Let $e_1, \ldots, e_n$ be an orthogonal basis of V with $F(e_i, e_j) = a_i \delta_{ij}$, $a_i \in k^*$. Set $D = \operatorname{diag}(a_1, \ldots, a_n)$. Then the condition $A \in GU(V, F)$ is equivalent to $A^\sigma D A^t = \lambda(A) D$. Replace A by a $(n + 1) \times (n + 1)$ matrix $\tilde{A} = A \oplus \lambda(A)^{-1}$. Then the condition $A^\sigma D A^t = \lambda(A) D$ can be written in the form

$\sum \tilde{a}_{ij}^{\sigma} a_j \tilde{a}_{kj} \tilde{a}_{n+1n+1} = a_i \delta_{ik}$ of polynomial k-relations on $\tilde{A}$ (since $a_i \in k$). This shows that $GU(V, F)$ can be identified with an algebraic k-subgroup of $R_{K/k}$ $GL(V)$. The assertions for U and SU are now clear.

1.3.6. It is easily shown that the center of $GU(V, F)$ is k-isomorphic to $R_{K/k}\mathbb{G}_{m,K}$. We define $PGU(V, F) = GU(V, F)/\text{center}$.

LEMMA. *$PGU(V, F)$ is a k-form of $PGL_k(n)$*

Proof. By extending k if necessary, we may assume that $K = k \oplus k$ and V has an orthonormal basis $e_1, \ldots, e_n$. Then $R_{K/k}\mathbb{P}_K^{n-1} \cong \mathbb{P}_k^{n-1} \times \mathbb{P}_k^{n-1}$ or, more explicitly, $\mathbb{P}(R_{K/k}V) = \mathbb{P}(uV) \times \mathbb{P}(u^{\sigma}V)$. We have, therefore, two projections $\psi_i : PGU(V, F) \to PGL_k(n)$, $i = 1, 2$, corresponding to the action of $PGU(V, F)$ on the first or second factor of $R_{K/k}\mathbb{P}_K^{n-1}$. Let $A \in GU(V, F)$ and $\bar{A}$ be its image in $PGU(V, F)$. If $\bar{A} \in \operatorname{Ker} \psi_1$ then $A = aI + uB$ where $a \in K^*$, I is the identity and B is a matrix with entries from uK. The condition that $A \in GU(V, F)$ becomes $A^{\sigma}A^t = bI$, whence $ua^{\sigma}B^t + u^{\sigma}aB^{\sigma} = (b - aa^{\sigma})I$. Since $uK \cap u^{\sigma}K = 0$ it follows that B is diagonal, say $B = \operatorname{diag}(b_1, \ldots, b_n)$. Then $\operatorname{Tr}(ua^{\sigma}b_i) = b - aa^{\sigma}$ and hence $a^{\sigma}b_i = u(b - aa^{\sigma})$. Thus B is a scalar matrix and A belongs to the center of $GU(V, F)$, that is $\bar{A} = \bar{I}$. Hence $\operatorname{Ker} \psi_1$ and similarly $\operatorname{Ker} \psi_2$ are trivial. It remains to show that ψ_1 and ψ_2 are epimorphisms. For this it suffices to show that the restriction of $GU(V, F)$ to uV contains $GL(uV)$. First $\operatorname{diag}(c_1u + c_1^{-1}u^{\sigma}, \ldots, c_nu + c_n^{-1}u^{\sigma})$, $c_i \in k^*$, restricts to $\operatorname{diag}(c_1, \ldots, c_n)$. Next, all the permutation matrices $uP + u^{\sigma}P$, where $P \in GL_k(n)$ is a permutation matrix, belong to $GU(V, F)$ since $(uP + u^{\sigma}P)^{\sigma}(uP + u^{\sigma}P)^t = (u^{\sigma}P + uP)(uP^{-1} + u^{\sigma}P^{-1}) = I$. Finally, the root subgroup $\{\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right) \oplus I_{n-2} \mid c \in k\}$ is in the restriction since

$$\begin{pmatrix} 1 & -u^{\sigma}c \\ uc & 1 \end{pmatrix} \oplus I_{n-2} \in GU(V, F)$$

restricted to uV is $\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right) \oplus I_{n-2}$.

1.3.7. PROPOSITION. *Let G be the adjoint group of $GU(V, F)$. Then the canonical map $GU(V, F)(k) \to G(k)$ is surjective.*

Proof. The group G acts on $R_{K/k}\mathbb{P}_K^{n-1}$ and preserves the duality F (cf. 1.2.1). Let $\bar{A} \in G(k)$ and let $A \in GL(V)$ be a preimage of $\bar{A}$ under the canonical projection. Since G preserves duality we have $F(Ax, Ay) = 0$ whenever $F(x, y) = 0$. Thus, by 1.3.4, $A \in GU(V, F)$ as asserted.

1.3.8. For a linear K-subspace M of $\mathbb{P}_K^{n-1}$ the groups $G^M = \{g \in PGU(V, F) \mid gM = M\}$ and $G_M = \{g \in G^M \mid g \mid M^{\perp} = 1\}$ are defined over k. Clearly, $G^{M^{\perp}} = G^M$.

If M has a regular preimage in V of rank m, then $G^M = G_M G_{M^{\perp}}$ and G^M is reductive with semisimple part of type $A_{m-1} \times A_{n-1-m}$.

1.3.9. Proposition. *$PGU(V, F)$ is split over k when $K = k \oplus k$. If K is a field then K is the smallest splitting field of $PGU(V, F)$.*

Proof. That $PGU(V, F)$ is split over k when $K = k \oplus k$ was established in the proof of 1.3.6 since it was shown explicitly that $PGU(V, F)$ is k-isomorphic to $PGL(uV)$. If K is a field then $\mathrm{Gal}(K/k)$ permutes the factors $\mathbb{P}^{n-1}$ in $R_{K/k}\mathbb{P}_K^{n-1}$ and none of them are defined over k. Therefore $PGU(V, F)$ is not split in this case. It is clear for $\tilde{K}$ a splitting field of $PGU(V, F)$ that $\tilde{K} \otimes_k K = \tilde{K} \oplus \tilde{K}$. Thus $\tilde{K}$ contains K and K is the smallest splitting field of $PGU(V, F)$.

1.3.10. A maximal k-subtorus T of $PGU(V, F)$ is called *admissible* if it is split over k if $K = k \oplus k$, or anisotropic and split over K if K is a field. A preimage of an admissible subtorus $T \subset PGU(V, F)$ in $GU(V, F)$ is also called admissible.

Proposition. *The admissible tori of $PGU(V, F)$ are exactly the pointwise stabilizers in $PGU(V, F)$ of n pairwise orthogonal points of $\mathbb{P}_K^{n-1}(K)$. The set of n pairwise orthogonal points is uniquely determined by the corresponding admissible torus.*

Proof. Let $L_1, \ldots, L_n$ be n lines in V such that $F(L_i, L_j) = K\delta_{ij}$. Let $\tilde{T}$ be the stabilizer fixing each of these lines in $GU(V, F)$. Then the action of $\tilde{T}$ on L_i defines a homomorphism $\kappa_i : \tilde{T} \to GU(L_i, F \mid L_i)$. Let $\lambda_i : GU(L_i, F \mid L_i) \to \mathbb{G}_{m,k}$ be the homomorphism λ of 1.3.4 for $GU(L_i, F \mid L_i)$. The product map $\prod \kappa_i$: $\tilde{T} \to \prod_i GU(L_i, F \mid L_i)$ is an imbedding and identifies $\tilde{T}$ with the subgroup $\{\{g_i\} \in \prod GU(L_i, F \mid L_i) \mid \lambda_i(g_i) = \lambda_j(g_j),\ 1 \leqslant i, j \leqslant n\}$. The group $GU(L_i, F \mid L_i)$ is k-isomorphic to $S = R_{K/k}\mathbb{G}_{m,K}$ and the map λ_i coincides on it with the norm map; set $\tilde{S} = \mathrm{Ker}(N_{K/k} : R_{K/k}\mathbb{G}_{m,K} \to \mathbb{G}_{m,k})$. Then $1 \to \tilde{S}^n \to \tilde{T}^\lambda \to \mathbb{G}_{m,k} \to 1$ is an exact sequence. The group $\tilde{S}$ is a k-form of $\mathbb{G}_{m,k}$; it is split over k if $K = k \oplus k$, and it is anisotropic over k and split over K if K is a field. Since the center of $GU(V, F)$ is k-isomorphic to S, it follows that the image of $\tilde{T}$ in $PGU(V, F)$ is isomorphic to $\tilde{T}/S$, that is, it is the image of $\tilde{S}^{n-1}$.

Conversely, suppose that T is an admissible torus in $PGU(V, F)$. Let $\tilde{T}$ be its preimage in $GU(V, F)$. Then $\tilde{T}$ is diagonalizable over k if $K = k \oplus k$ and over K if K is a field. Suppose first that $K = k \oplus k$. Since all maximal k-tori split over k are conjugate over k it follows that we can assume that $\tilde{T}$ is the torus described in the previous paragraph. Thus there exist n pairwise orthogonal lines $L_1, \ldots, L_n$ such that $\tilde{T}L_i \subseteq L_i$. Then each L_i affords two k-characters of $\tilde{T}$, say χ_i and $\bar{\chi}_i$. If K is a field then in $V \otimes_k K$ there exist lines L_i affording k-characters χ_i and $\bar{\chi}_i$. We easily establish that $\chi_i^\tau = \chi_i$ for $1 \neq \tau \in \mathrm{Gal}(K/k)$ whence it follows that L_i is defined over k, that is, $L_i = \tilde{L}_i \otimes K$ for $\tilde{L}_i$-lines in V.

Remark. Let $\tilde{T}$ be the preimage in $SU(V, F)$ (resp. $U(V, F)$) of an admissible torus of $PGU(V, F)$. Then the argument of the first paragraph of the proof shows that $\tilde{T} = \tilde{S}^{n-1}$ (resp. $\tilde{S}^n$).

1.3.11. Let T be an admissible torus of $G = PGU(V, F)$ and let $\Sigma(G, T)$ be the root system of G with respect to T. For each $a \in \Sigma(G, T)$ let G_a be the three-dimensional subgroup of G of type A_1 normalized by T which affords the roots $\pm a \in \Sigma(G, T)$. An admissible torus $\tilde{T}$ of G is called *associated* with T if $\tilde{T} \subset G_a T$ for some $a \in \Sigma(G, T)$. The following is a direct consequence of 1.1.4 and 1.3.10 (compare also with [14]).

PROPOSITION. *Except when* $|k| = 2$, *for any pair* T, $\tilde{T}$ *of admissible tori of* G *there exists a sequence* $T_0, T_1, \ldots, T_m$ *of admissible tori such that*

(i) $T_0 = T$, $T_m = \tilde{T}$ *and* T_i *is associated with* T_{i-1},

(ii) $m \leqslant \frac{1}{2}n(n-1)$ *if* char $k \neq 2$ *or if* F *is anisotropic,*

(iii) $m \leqslant 4$ *when* $n = 3$.

2. A Completion of a Set of Tori

We preserve the notation K, V, F, $U(V, F)$, etc. from Section 1. Set $G = PGU(V, F)$ and denote by ω the canonical map ω: $GU(V, F) \to PGU(V, F)$. Let X be the set one-dimensional of tori of G whose centralizer $Z_G(T)$ is a group of type A_{n-2} (actually isomorphic to $GL(n-1)$) and by Y the set of (unipotent one-dimensional) root subgroups of G. In the terminology of [7] the elements of $T \in X$ are (projective) dilations and the elements of $U \in Y$ are (projective) transvections. Both X and Y are defined over k. We are going to identify $X \cup Y$ with $\mathbb{P}_K^{n-1}$ and show how the geometry of $\mathbb{P}_K^{n-1}$ can be expressed in terms of the group G.

Assume from now on that $n = rk_K V \geqslant 3$.

2.1. *Splitting Properties of k-tori T in X*

2.1.1. LEMMA. *If* T *is a* k*-subtorus of* G, $T \in X$, *then the smallest splitting field of* T *is also the smallest splitting field of* G.

Remark. Since dim $T = 1$ the smallest splitting field of T has degree at most two over k.

Proof. Let P_1, P_2 be the two maximal opposite parabolic subgroups of G such that $P_1 \cap P_2 = Z_G(T)$. If T is split over k then P_1 and P_2 are defined over k. This shows that the smallest splitting field of T splits G. Now suppose that G is split. Then the varieties G/P_1 and G/P_2 are defined over k. But if T were not split then the Galois group of the smallest splitting field of T would permute P_1 and P_2 in contradiction with the fact that G/P_1 is defined over k.

2.1.2. COROLLARY. *A* k*-torus* $T \in X$ *is contained in an admissible torus.*

Proof. If T is split over k then T is k-conjugate to a subtorus of an admissible torus. If T is not split over k, then the group $Z_G(T)$ is split over the smallest splitting field of T by 2.1.1 and therefore $Z_G(T)$ contains an admissible torus of G containing T.

2.1.3. COROLLARY. *Let $T \in X$ be defined over k. Then $T = G_L$ and $Z_G(T) = G_{L^\perp}$ for a unique regular point $L \in \mathbb{P}_K^{n-1}(K)$.*

Proof. Let $\tilde{T}$ be an admissible subtorus of G contained in $Z_G(T)$ (it exists since T and therefore $Z_G(T)$ are split over the smallest splitting field of G). Let $L_1, \ldots, L_n \in \mathbb{P}_K^{n-1}(K)$ be the unique set of n pairwise orthogonal lines such that $\tilde{T}L_i = L_i$ (see 1.3.10). Then the groups G_{L_i} are defined over k, contained in $\tilde{T}$ and belong to X. On the other hand $\tilde{T}$ contains exactly n subtori from X. Thus T must coincide with one of G_{L_i}, as asserted.

2.2. *Geometric Interpretation of X*

2.2.1. LEMMA. *Let L be a regular point of $\mathbb{P}_K^{n-1}(K)$. Then $T = G_L$ is a torus in $X(k)$.*

Proof. This follows from 1.3.8.

2.2.2. Thus the tori $T \in X(k)$ are in one-to-one correspondence with the regular points $L \in \mathbb{P}_K^{n-1}(K)$; all of them are defined over k. Namely, to $T \in X(k)$ there corresponds a unique fixed point $L(T) \in \mathbb{P}_K^{n-1}(K)$ of $Z_G(T)$, while to $L \in \mathbb{P}_K^{n-1}(K)$ there corresponds the torus $T(L) = Z_G(DG_{L^\perp}) \in X(k)$.

2.2.3. PROPOSITION. *A torus $T \in X$ has the following properties.*

(i) $Z_G(t) = Z_G(T)$ *for any* $t \in T$, $t \neq 1$,

(ii) $N_G(T) = Z_G(T)$,

(iii) $T = C(Z_G(T))$,

(iv) $T = Z_G(DZ_G(T))$,

(v) $DZ_G(T)$ *is absolutely almost simple.*

Proof. These can be easily verified.

2.3. *Geometric Interpreation of Y*

2.3.1. LEMMA. *Let L be an isotropic line of V. Then the image $U(L)$ in G (under ω: $GU(V, F) \to G$) of the group of transvections $x \mapsto x + aF(x, v)v$, where $Kv = L$ and* $\mathrm{Tr}(a) = 0$, *is a root k-subgroup.*

Proof. Since F is nondegenerate v is contained in a regular plane of V. Every line of this plane is stable under a k-subtorus of G of dimension $n - 2$, so it suffices to show that $U(L)$ is normalized by a maximal k-torus only in the case where V is a plane. We may assume that $V = Kv + Kw$ with $F(v, w) = 1$

and $f(w) = 0$. Then the maximal torus $T = \{v \mapsto bv,\ w \mapsto b^{-1}w \mid b \in k^*\}$ in $SU(V, F)$ is defined and split over k. Since it leaves Kv stable, ωT normalizes $U(L)$. That $U(L)$ is defined over k follows from the fact that it is normalized by a maximal torus defined over k (cf. the proof of 8.2 in [2]).

2.3.2. LEMMA. *Let U be a root k-subgroup of G. Then there exists a unique point $L = L(U) \in \mathbb{P}_K^{n-1}(K)$ fixed by $\omega^{-1}(Z_G(U))$. Moreover, $U(L(U)) = U$. In particular, U is k-isomorphic to $\mathbb{G}_{a,k}$.*

Proof. Let $\tilde{L}$ be an isotropic point of $\mathbb{P}_K^{n-1}(K)$. Set $P = N_G(U)$ and $P_1 = N_G(U(\tilde{L}))$. If k is perfect then P and P_1 are k-conjugate and it is easily checked that the lemma holds for groups of the type in 2.3.1. If k is imperfect then it is infinite. Since P and P_1 are $\bar{k}$-conjugate there exists $g \in G(k)$ such that P is opposite to gP_1g^{-1}. Set $\tilde{H} = P \cap gP_1g^{-1}$. Then $D\tilde{H}$ is semisimple of type A_{n-3} and $H = DZ_G(D\tilde{H})$ is $\bar{k}$-isomorphic to $SL(2)$. Then $H = \langle U, gU(\tilde{L})g^{-1}\rangle$ and since both U and $gU(\tilde{L})g^{-1} = U(g\tilde{L})$ are defined over k, it follows that H is defined over k. Since H is a k-form of $SL(2)$ it follows (using, for example, an identification with elements of norm 1 in a quaternion algebra) that every unipotent k-subgroup of H is contained in a Borel k-subgroup. Thus all such subgroups are k-conjugate. In particular, $U = hU(\tilde{L})h^{-1} = U(h\tilde{L})$ for some $h \in G(k)$. This shows the existence of L. The other statements can be explicitly checked for the groups $U(L)$ in 2.3.1.

2.3.3. Thus the groups $U \in Y(k)$ are in one-to-one correspondence with isotropic points $L \in \mathbb{P}_K^{n-1}(K)$. The correspondence is given by $L \mapsto U(L)$, $U \mapsto L(U)$.

2.3.4. PROPOSITION. *A group $U \in Y$ has the following properties.*

(i) $Z_G(U) = Z_G(u)$ *for any* $u \in U$, $u \neq 1$,

(ii) $N_G(U)/Z_G(U) \cong \mathbb{G}_{m,k}$,

(iii) $U = C(Z_G(U))$,

(iv) $U = Z_G(DZ_G(U))$, *moreover, if* $n = 3$ *then* $D^2Z_G(U) = U$ *and* $D^3Z_G(U) = 1$, *while if* $n > 3$ *then* $D^2Z_G(U) = DZ_G(U)$.

Proof. These can be easily verified.

2.4. *The Algebraic Geometry of $X \cup Y$*

2.4.1. THEOREM. *$X \cup Y$ carries a natural structure of an algebraic k-variety isomorphic to $R_{K/k}\mathbb{P}_K^{n-1}$. This structure coincides with existing structures on X and Y.*

Proof. We established in 2.2.2 and 3.3.3 that there exists a bijection φ: $(X \cup Y)(k) \to (R_{K/k}\mathbb{P}_K^{n-1})(k)$. It is given by $\varphi(H) = L$, the unique fixed point of $Z_G(H)$ in $\mathbb{P}_K^{n-1}$, and $\varphi^{-1}(L) = Z_G(DG^L)$ for $H \in X \cup Y$, $L \in \mathbb{P}_K^{n-1}$. It is clear from the construction that this identification makes $X \cup Y$ into an algebraic variety defined over k and that its restrictions to X and Y are identifications with the quotient spaces of G ($G/Z_G(T)$, $T \in X$, and $G/N_G(U)$, $U \in Y$, for X and Y respectively).

2.4.2. Let Z_1 and Z_{n-1} be the varieties of the maximal parabolic subgroups corresponding to the extreme nodes of the Dynkin diagram of G. So $Z_1 \cong \mathbb{P}^{n-1}$, $Z_{n-1} \cong \mathbb{P}^{n-1}$ and the structures of the projective spaces on Z_1 and Z_{n-1} are dual. The variety $Z_1 \times Z_{n-1}$ is defined over k.

THEOREM. *There exists a natural k-isomorphism* $\tilde{\varphi}\colon Z_1 \times Z_{n-1} \to X \cup Y$.

Proof. This can be easily deduced from a k-isomorphism $Z_1 \times Z_{n-1} \to R_{K/k}\mathbb{P}_K^{n-1}$ which can be easily verified if $K = k \oplus k$ and follows in general by Galois descent. We give a direct proof for $K = k \oplus k$ using a natural realization of $GL_k(n)$. For simplicity of notation assume that $G = GL(\tilde{V})$ where $\tilde{V} = u^\circ V$ (cf. 1.3.6). Then $P \in Z_1$ is the stabilizer of a flag $\tilde{V} \subset V_1$, $\dim V_1 = 1$, in $\tilde{V}$ and $P' \in Z_{n-1}$ is the stabilizer of a flag $\tilde{V} \supset V_{n-1}$, $\dim V_{n-1} = n-1$, in $\tilde{V}$. If $\tilde{V} = V_1 \oplus V_{n-1}$ then $\tilde{\varphi}(P, P') = Z_G(D(P \cap P'))$ consists of linear mappings acting as scalars on V_1 and V_{n-1}. If $V_{n-1} \supset V_1$ then $\tilde{\varphi}(P, P') = Z_G(D(P \cap P'))$. In this case consider a unique (up to a scalar) linear form l on V such that $l \neq 0$ and $l(V_{n-1}) = 0$. Then $\tilde{\varphi}(P, P')$ consists of transvections $x \mapsto x + l(x)v$, $v \in V_1$. Thus we have our bijective map and it is routine to check that it is a k-isomorphism on X and Y. It follows that it is a k-isomorphism on $X \cup Y$.

2.4.3. *Remark.* If $n = 2$ then $Z_1 = Z_{n-1}$ and the set $X \cup Y$ is now isomorphic to the symmetric square of Z_1, that is, $X \cup Y \cong S^2(\mathbb{P}^1) \cong P^2$. The reason is that for $T \in X$ we have that $T = Z_G(T)$ fixes two points on $\mathbb{P}_K^{n-1}$. That is $[N_G(T)\colon Z_G(T)] = 2$ and therefore $\tilde{\varphi}(P, P') = \tilde{\varphi}(P', P)$.

Our considerations do describe subgroups G_L of G, $L \in \mathbb{P}_K{}^1(K)$ when $n = 2$. In this case we have the map $\mathbb{P}_K{}^1 \to \mathbb{P}_k{}^2 = S^2\mathbb{P}_k{}^1$ and the subgroups G_L, $L \in \mathbb{P}_K{}^1(K)$ correspond to the image of $\mathbb{P}_K{}^1(K)$ in $\mathbb{P}^2(k)$. As we know from [13], $\mathbb{P}^2(k)$ parametrizes connected one-dimensional k-subgroups. Also, from [13] not all such subgroups are of the form G_L, $L \in \mathbb{P}_K{}^1(K)$ when char $k = 2$. This corresponds to the fact that if char $k = 2$, the image of $\mathbb{P}_K{}^1(K)$ does not cover all of $\mathbb{P}^2(k)$. Preimages of some points from $\mathbb{P}^2(k)$ are defined over a quadratic inseparable extension and become rational only after a map $\mathbb{P}_K{}^1 \to \mathbb{P}_k{}^2$ of degree two.

2.4.4. PROPOSITION. *Let Lie*: $H \mapsto$ *Lie* H *be the mapping from the set of subgroups of G to the set of Lie subalgebras of Lie G. Then the closure of the set* $\{\mathrm{Lie}\ T \mid T \in X\}$ *in the Grassmanian* $G_{1,\dim G} \cong \mathbb{P}_k^{\dim G-1}$ *of* 1*-dimensional subspaces of Lie G is k-isomorphic to* $R_{K/k}\mathbb{P}_K^{n-1}$.

Proof. One can verify the Proposition directly using the same method as in the proof of 2.4.3, or one can use the following approach. For adjoint groups of type A_n the map *Lie* is bijective, it induces k-isomorphisms on X and Y and commutes with the action of G. It can be checked that the closure of X contains Y but cannot (for dimension reasons) contain any other orbits of G. Therefore the closure is $X \cup Y$. The coincidence of algebraic k-structures is then automatic.

2.4.5. *Remark.* (i) The statement 2.4.4 is false for forms of $\mathrm{SL}(n)$ since the map *Lie* sends all tori $T \in X$ of $\mathrm{SL}(p)$ to the center of Lie $\mathrm{SL}(p)$ if char $k = p$.

(ii) One can prove an analog of 2.4.4 for the group G itself. To do this one must have a notion of Hilbert or Chow schemes of subvarieties of G. To define this we compactify G in the following way. Consider the map $G \to G/B$ where B is a Borel subgroup of G. Its fibers are $T \times N$ where $N = [B, B]$. Consider the associated fiber space to $G \to G/B$ with the fiber $\hat{T} \times G/B$ where $\hat{T}$ is a completion of T and the action of N on G/B agrees with the "big cell" imbedding $N \subset G/B$. For $\hat{T}$ we can take the completion of Mumford [10]. Note that $G \to G/B$ can be represented as the composite of two maps $G \to G/U$ and $G/U \to G/B$ with fibers U and T respectively. Since G acts on G/B the fiber space associated to $G \to G/U$ is actually a direct product. Thus our completion is

$$G/B \times [\text{the fiber space associated to } G/U \to G/B \text{ with fiber } \hat{T}].$$

This completion was found by D. Kazhdan.

2.5. *Group Interpretation of the Autoduality*

We want to express the orthogonality of lines in V in group theoretic terms. Let the groups $H_1, H_2 \in X \cup Y$, $H_1 \neq H_2$, correspond to the lines $L_1 = L(H_1)$, $L_2 = L(H_2)$ in V, where $L_1 \neq L_2$.

PROPOSITION. *L_1 is orthogonal to L_2 if and only if* $[H_1, H_2] = 1$.

Proof. It is easily seen that $F(L_1, L_2) \neq 0$ if and only if the two groups do not commune.

2.6. *Group Interpretation of Lines of $X \cup Y$*

Let $\mathscr{P}$ be the set of lines of $X \cup Y \simeq \mathbb{P}_K^{n-1}$. We know by 1.1.6 that $\mathscr{P}$ is the disjoint union $\mathscr{P} = \mathscr{P}_r \cup \mathscr{P}_d \cup \mathscr{P}_i$ where $\mathscr{P}_r$ (resp. $\mathscr{P}_d$; resp. $\mathscr{P}_i$) is the set of regular (resp. degenerate, but not totally isotropic; resp. totally isotropic) lines of $\mathbb{P}_K^{n-1}$ (or planes of V). For each point L in $\mathbb{P}_K^{n-1}(K)$, let $H(L)$ be the corresponding group in $X \cup Y$ (cf. 2.2.2 and 2.3.3) and for each line P of

$\mathbb{P}_K^{n-1}$ denote by $H(P)$ the group generated by the groups $H(L)$ with $L \in P$. Our aim is to describe these groups $H(P)$ for $P \in \mathscr{P}_r$, $\mathscr{P}_d$, $\mathscr{P}_i$.

2.6.1. Lemma. *Let P and L be respectively a line and a point of $\mathbb{P}_K^{n-1}$. Then $F(P, L) = 0$ if and only if $[H(P), H(L)] = 1$.*

Proof. This is an immediate consequence of 2.5 and the definition of $H(P)$.

2.6.2. Lemma. *If M is a regular subspace of V then the subgroup of G generated by the groups $H(L)$, L a line in M, is the group G_M.*

Proof. This follows from the definition of G_M in 1.3.8.

2.6.3. Lemma. *If $P \in \mathscr{P}_d$ then*

(i) *$H(P)$ is contained in G_M where M is a regular 3-dimensional subspace of V,*

(ii) *$H(P)$ is conjugate in $G_M \cong GL(3)$ to the subgroup of matrices of the form*

$$\begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & a \end{pmatrix},$$

(iii) *$DH(P)$ is unipotent and $D^2H(P) = H(P \cap P^\perp)$, where $P \cap P^\perp$ is an isotropic line.*

Proof. (i) Follows from 1.1.6; (ii) is verified directly; (iii) follows from (ii).

2.6.4. Lemma. *If $P \in \mathscr{P}_i$ then*

(i) *$H(P)$ is contained in G_M where M is a regular 4-dimensional subspace of V,*

(ii) *$H(P)$ is the unipotent radical of a parabolic subgroup of $GL(4)$ whose semisimple part is of type $A_1 \times A_1$. Explicitly, $H(P)$ is conjugate in $GL(4)$ to a subgroup of matrices of the form*

$$\begin{pmatrix} 1 & 0 & a & b \\ & 1 & c & d \\ & 0 & 1 & 0 \\ & & & 1 \end{pmatrix},$$

(iii) *$H(P)$ is commutative.*

Proof. (i) follows from 1.1.6; (ii) is verified directly; (iii) follows from (ii).

2.6.5. Proposition. *A subgroup H of G is of the form $H(P)$ for P a line of*

$\mathbb{P}_K^{n-1}(K)$ *if and only if there exists an admissible torus* $T \subset G$ *with* $T_1, T_2, \ldots, T_n$ *the set of elements of* $X(k)$ *belonging to* T *such that one of the following holds*:

(i) $H = Z_G Z_G D Z_G(T_1, \ldots, T_{n-2})$;

(ii) $H = Z_G(U, Z_G D Z_G(T_1, \ldots, T_{n-3}))$ *where* $U \in Y(k)$ *and* $U \subset DZ_G(T_1, \ldots, T_{n-3})$;

(iii) (*only for* $n \geqslant 4$) $H = Z_G(U_1, U_2, Z_G D Z_G(T_1, \ldots, T_{n-4}))$ *where* $U_1, U_2 \in Y(k)$, $U_1 \subset DZ_G(T_1, \ldots, T_{n-4})$, $U_2 \subset DZ_G(T_1, \ldots, T_{n-2})$ *and* $[U_1, DZ_G(T_1, \ldots, T_{n-2})] = 1$.

Proof. The three parts correspond to 2.6.2, 2.6.3 and 2.6.4, respectively, and can be verified by matrix computation. For example, the elements of $Z_G(T_1, \ldots, T_{n-2})$ can be viewed as matrices of the form

$$\left(\begin{array}{ccc|cc} a_1 & & 0 & & \\ & \ddots & & & 0 \\ 0 & & a_{n-2} & & \\ \hline & 0 & & * & * \\ & & & * & * \end{array}\right)$$

in G, and those in $Z_G Z_G D Z_G(T_1, \ldots, T_{n-2})$ are then of the form

$$\left(\begin{array}{c|cc} aI_{n-2} & & 0 \\ \hline 0 & * & * \\ & * & * \end{array}\right),$$

and thus correspond to $H(P)$ with P a regular plane.

2.6.6. *Remark.* Let $\tilde{Y}$ be the set of subgroups $H(P)$ of G where P varies over regular planes of V (or regular lines of $\mathbb{P}_K^{n-1}(K)$). Then 2.6.5(i) shows that the closure of $\tilde{Y}$ (in the sense, say, of 2.4.5(ii)) is the Grassmanian $G_{2,n,K}$ of two-planes in V (cf. [6, I, Sect. 1.3.4]). Then 2.6.5(i) says that the relation of inclusion between $\mathbb{P}_K^{n-1}$ and $G_{2,n,K}$ can be expressed in group terms of the algebraic group G.

3. Full Groups

We introduce now the full subgroups of $G = PGU(V, F)$ in the sense of O'Meara. However, for our purposes it is necessary to also develop a more axiomatic characterization.

Recall that $n = rk_K V \geqslant 3$ and we now also assume that k is an infinite field.

3.1. *Density Properties of Full Subgroups*

3.1.1. DEFINITION. A subgroup Q of $G(k)$ is *full* if for every $H \in (X \cup Y)(k)$ one has $|H \cap Q| > 1$. That is, for every line L of V with corresponding group $H(L)$ in $X \cup Y$ we have $|H(L) \cap Q| > 1$. Set $Q(L) = H(L) \cap Q$ for $L \in \mathbb{P}_K^{n-1}(K)$ and denote by X_Q (resp. Y_Q) the set of subgroups $Q(L)$ of Q with L a regular (resp. isotropic) line.

3.1.2. *Remark.* It is enough to assume that $|T \cap Q| > 1$ for $T \in X(k)$. For if we consider a plane $P \in \mathscr{P}_d$, then (cf. 2.6.3) $U(P \cap P^{\perp}) \cap Q \supseteq D^2(\langle T(L) \cap Q : L$ a regular line in $P\rangle) \neq 1$ since $\langle T(L) \cap Q\rangle$ is dense in $H(P)$. It follows that $|U(L) \cap Q)| > 1$ for isotropic lines L, that is, $|U \cap Q| > 1$ for $U \in Y(k)$.

3.1.3. EXAMPLES. Call a subgroup $R \subseteq G(k)$ *very full* if $|R \cap T| = \infty$ for all $T \in X(k)$. Trivially, very full subgroups are also full. Then the following statements can be easily verified.

(a) $G(k)$ is very full;

(b) if ω: $\tilde{G} \to G$ is a surjective map and $\tilde{R}$ is very full in $\tilde{G}(k)$, then $\omega(\tilde{R})$ is very full;

(c) if R is very full, then $R(n) = \langle r^n : r \in R\rangle$ is very full (in particular, subgroups of finite index in very full groups are very full);

(d) if R is full and S is a subgroup of $G(k)$ containing R, then S is full.

3.1.4. To study full groups we need to record some standard properties of dense subgroups.

LEMMA. *Let G be a connected algebraic k-group and G_1, G_2,..., G_m closed algebraic k-subgroups of G. Let H be a dense subgroup of G and let H_i be a dense subgroup of G_i.*

(i) *If H_1 is normal in H then G_1 is normal in G and $H/H \cap G_1$ is dense in G/G_1;*

(ii) *$Z_G(H_1) = Z_G(G_1)$ and $Z_H(H_1) = Z_G(G_1) \cap H$;*

(iii) *D^iH is dense in D^iG;*

(iv) *if G is connected and reductive then D^iH is dense in DG for all $i \geqslant 1$; in particular, if G is not a torus then $D^iH \neq 1$ for $i \geqslant 1$;*

(v) *if G_1 is connected and reductive then $Z_H(D^iH_1) = Z_H(DH_1) = H \cap Z_G(DG_1)$;*

(vi) *$\langle H_1$,..., $H_m\rangle$ is dense in $\langle G_1$,..., $G_m\rangle$;*

(vii) *if G is not unipotent, H contains a regular element of G of arbitrary large order.*

Proof. In (i) it is clear that G_1 is normal in G. Let $\bar{G}$ be the closure of HG_1/G_1 in G/G_1. Then the preimage of $\bar{G}$ contains H whence $\bar{G} = G/G_1$. The assertion (ii) is clear. Now (iii) follows from (i) if we take G_1 to be the closure of $H_1 = DH$. Next, (iv) holds because $DG = D^iG$ for $i > 1$ and because of (iii). Now (v) is the combination of (ii) and (iv). To prove (vi) note that $H_{i_1} \times H_{i_2} \times \cdots \times H_{i_m}$ is dense in $G_{i_1} \times G_{i_2} \times \cdots \times G_{i_m}$ whence the image of the first product in G is dense in the image of the second. Finally, (vii) holds since the set of regular elements of order $\geqslant n_0$ is open in G and non-empty for any $n_0 \in \mathbb{Z}$.

3.1.5. PROPOSITION. *For every regular subspace M of V and full subgroup Q of $G(k)$, the group $D(Q(M))$, where $Q(M) = \langle Q \in T(L)\colon L$ a regular line in $M\rangle$, is dense in DG_M.*

Proof. This a consequence of the definitions and 2.6.2.

3.1.6. DEFINITION. Call a group S *fop-simple* (that is, simple For Our Purposes) if for every non-central subgroup S_1 of S which is normalized by a non-central normal subgroup of S we have

(i) $D^mS_1 \neq 1$ for $m \geqslant 0$;

(ii) if S_1 contains two commuting normal subgroups then at least one of them is central.

3.1.7. LEMMA. *Let R be a connected almost k-simple algebraic k-group and let S be a dense subgroup of $R(k)$. Then*

(i) *the center of S is finite and is contained in the center of R;*

(ii) *S is fop-simple.*

Proof. The first assertion is clear. Let us prove (ii). Let N be a non-central normal subgroup of S. By 3.1.4(i) its Zariski closure is a non-central normal subgroup of R. It is defined over k since $N \subseteq R(k)$. Since R is almost k-simple it follows that N is dense in R. Let S_1 be a non-central subgroup of S normalized by N. Then by 3.1.4(i) S_1 is dense in R. It then follows from 3.1.4(iii) that D^mS_1 is dense in R for all m so that $D^mS_1 \neq 1$. Suppose that S_1 has two commuting normal subgroups, say N_1 and N_2. Then their Zariski closures are commuting normal k-subgroups of R and hence these closures are either central or the whole of R. Since both these closures cannot be R at least one of them is central.

3.1.8. DEFINITION. The group S is called *quasi-fop-simple* if $S/$center is fop-simple.

3.2. *Full Triples*

We axiomatize below some properties of full subgroups of G. However, the list of properties given is not intended to be minimal. Controlling the choice of properties was the desire to make later proofs in Section 4 simpler and to make the verification that full subgroups of G do have the listed properties easier.

3.2.1. Let Q be an infinite group with two distinguished families of non-trivial subgroups: X_Q and Y_Q. For a subgroup $R \subseteq Q$ denote by $X_{Q,R}$ and $Y_{Q,R}$ the sets of subgroups from the corresponding families which belong to R. For a quotient $\bar{R}$ of R denote by $X_{Q,\bar{R}}$ and $Y_{Q,\bar{R}}$ the sets of non-trivial images in $\bar{R}$ of subgroups from $X_{Q,R}$ and $Y_{Q,R}$.

3.2.2. Now impose on subgroups $N \in X_Q$ the following conditions (cf. 2.2.3).

(i) $Z_Q(N) = Z_Q(h)$ for any $h \neq 1$ in N,

(ii) $N = C(Z_Q(N))$,

(iii) $N = Z_Q D^m Z_Q(N)$ for any m,

(iv) $DZ_Q(N)$ is quasi-fop-simple.

We say that the triple (Q, X_Q, Y_Q) is *good* if it satisfies the above conditions.

3.2.3. Let (Q, X_Q, Y_Q) be a good triple. A family of subgroups $N_1, \ldots, N_n \in X_Q$ is called a *chain* (of length n) if the following conditions are satisfied.

(i) $[N_i, N_j] = 1$ for $i, j \in [1, n]$,

(ii) if $I, J \subset [1, n]$, $|I| \leqslant n-2$, $|J| \leqslant n-2$ then an inclusion $D^r Z_Q(N_i, i \in I) \subset D^s Z_Q(N_j, j \in J)$ implies that $I \supset J$,

(iii) if $I \subset [1, n]$, $|I| \leqslant n-2$ then $Z_Q(N_i, i \in I)$ is quasi-fop-simple,

(iv) if $I \subset [1, n]$, $2 \leqslant |I| \leqslant n-2$ then $DZ_Q(D^r Z_Q(N_i, i \in I)) = DZ_Q(N_i, i \notin I)$ for any $r \geqslant 1$,

(v) if $I_1, \ldots, I_m$ are pairwise disjoint subsets of $[1, n]$ with $2 \leqslant |I_j| \leqslant n-2$ then the groups $H_s = DZ_Q(N_i, i \notin I_s)$ commute.

3.2.4. We say that a good triple (Q, X_Q, Y_Q) is *very good* (of rank $n-1$) if

(i) it has chains of length $n \geqslant 3$,

(ii) any chain is of length n,

(iii) any $N \in X_Q$ can be included in a chain,

(iv) if $n \geqslant 3$ consider $R = Z_Q(N)$ for $N \in X_Q$ and set $\bar{R} = R/\text{center}$; then the triple $(\bar{R}, X_{Q,\bar{R}}, Y_{Q,\bar{R}})$ (cf. 3.2.1) is a very good triple of rank $n-2$ (except, for $n = 3$, (i) is omitted) and all its chains are images of chains of (Q, X_Q, Y_Q) containing N.

Remark. The last condition is recursive but poses no logical danger since it is of finite length (namely, $n-2$).

3.2.5. Let us now impose conditions on Y_Q (cf. 2.3.4). We call a triple (Q, X_Q, Y_Q) *excellent* if it is very good of rank $n-1$ and for every $U \in Y_Q$ there exists a chain $N_1, \ldots, N_n$ such that

(i) U is commutative and $U = Z_Q DZ_Q(U)$,

(ii) $U \subset R$ and $Z_R(U)$ is commutative, where $R = Z_Q Z_Q DZ_Q(N_1, \ldots, N_{n-2})$,

(iii) $Z_Q(U) = Z_Q(u)$ for any $u \in U$, $u \neq 1$,

(iv) $Z_S(U)$ is solvable, $CZ_S(U) = U$, $DZ_S(U)$ is nilpotent and $1 \neq D^2 Z_S(U) \subseteq U$, where $S = DZ_Q(N_1, \ldots, N_{n-3})$.

3.2.6. Define a *regular line* of $X_Q \cup Y_Q$ as the set of subgroups from $X_Q \cup Y_Q$ contained in $Z_Q Z_Q DZ_Q(N_1, \ldots, N_{n-2})$ for some chain $N_1, \ldots, N_n$ of Q (cf. 2.6.5(i)). The set of regular lines is denoted by $\mathscr{P}_r(Q)$. Also define a *degenerate, but not totally isotropic line* of $X_Q \cup Y_Q$ as the set of subgroups from $X_Q \cup Y_Q$ contained in $Z_Q(U, Z_Q DZ_Q(N_1, \ldots, N_{n-3}))$ for some chain $N_1, \ldots, N_n$ and $U \in Y_Q$ with $U \subset DZ_Q(N_1, \ldots, N_{n-3})$. The set of such lines is denoted $\mathscr{P}_d(Q)$. When $n \geqslant 4$ we define a *totally isotropic line* of $X_Q \cup Y_Q$ as the set of subgroups from Y_Q contained in $Z_Q(U_1, U_2, Z_Q DZ_Q(N_1, \ldots, N_{n-4}))$ for some chain $N_1, \ldots, N_n$ and U_1 and U_2 in Y_Q with $U_1 \subset DZ_Q(N_1, \ldots, N_{n-4})$, $U_2 \subset DZ_Q(N_1, \ldots, N_{n-2})$, and $[U_1, DZ_Q(N_1, \ldots, N_{n-2})] = 1$. The set of such lines is denoted $\mathscr{P}_i(Q)$. A *line* is any one of the subsets from $\mathscr{P}(Q) = \mathscr{P}_r(Q) \cup \mathscr{P}_d(Q) \cup \mathscr{P}_i(Q)$.

3.2.7. We call (Q, X_Q, Y_Q) a *full triple* if it is an excellent triple and

(i) there exists an algebra K that is separable and of dimension two over some field k such that $X_Q \cup Y_Q$ with lines defined as in 3.2.6 is the projective space $\mathbb{P}_K^{n-1}(K)$ over K.

(ii) the orthogonality relation on $X_Q \cup Y_Q$, where H_1 and H_2 in $X_Q \cup Y_Q$ are orthogonal if and only if $[H_1, H_2] = 1$, determines an autoduality of $\mathbb{P}_K^{n-1}(K)$ connected with a non-degenerate K/k-Hermitian form.

3.2.8. THEOREM. *Let Q be a full subgroup of G with X_Q and Y_Q defined as in* 3.1.1. *Then (Q, X_Q, Y_Q) is a full triple.*

Proof. This (Q, X_Q, Y_Q), of course, has been our model for the definition of a full triple and most of the required properties have essentially been established in 2.2, 2.3, 2.5 and 2.6. The remaining properties can be verified in a similar manner. The main point is that a single nontrivial element in a subgroup H from $X_Q \cup Y_Q$ determines the centralizer of H. Also, for every element in $(X \cup Y)(k)$ there corresponds an element in $X_Q \cup Y_Q$.

4. MONOMORPHISMS OF UNITARY GROUPS

4.1. *Imbeddings of Full Triples in $PGL(m)$*

4.1.1. THEOREM. *Let k' be a field and $G' = PGL_{k'}(m)$. Let (Q, X_Q, Y_Q) be a full triple of rank $n - 1$. If $m \leqslant n$ and $Q \subseteq G'(k')$ then*

(i) $m = n$,

(ii) $Z_{G'}DZ_{G'}(N) \in X' = X(G')$ *for every* $N \in X_Q$.

Remark. Here, as before, we work mainly with groups from X_Q. This is because Y_Q may be empty (if Q is a full subgroup of G and G is anisotropic over k then $Y_Q = \varnothing$).

The proof will be broken into a number of steps. We may assume that k' is algebraically closed.

4.1.2. Let $N_1, \ldots, N_n$ be a chain and I_j, $1 \leqslant j \leqslant [n/2]$ be pairwise disjoint subsets of $[1, n]$ with $|I_j| \geqslant 2$. Set $H_s = DZ_Q(N_i, i \notin I_s)$. Then the groups H_s are quasi-fop-simple and pairwise commute by 3.2.3(iii), (v). Let $\bar{H}_s$ be the connected component of the Zariski closure of H_s in G'. Then $\bar{H}_s$ is the Zariski closure of a subgroup of finite index in H_s and therefore the groups $\bar{H}_s$ pairwise commute and are not solvable.

We want to prove first that the elements of N_i are semisimple.

4.1.3. Consider first the case of even n. Then since $m \leqslant n$ and since $PGL(m)$ can contain at most $[m/2]$ commuting non-solvable groups it follows that $m = n$ and that all the groups $\bar{H}_s$ are of type A_1 and are normalized by a maximal torus. It follows from 3.2.3(ii) that $G'' = Z_{G'}DZ_{G'}(N_1, N_2, N_3, N_4)$ is of type A_3 (actually, it is G' if $n = 4$ and it is isomorphic to $GL(4)$ if $n \geqslant 6$) and by 3.2.4(iv) the image of N_1, N_2, N_3, N_4 in $Q'' = Z_QDZ_Q(N_1, N_2, N_3, N_4)/\text{center}$ is a chain in Q''. Since the center of G' is semisimple it is sufficient (for the proof of semisimplicity of N_i) to consider the case $n = 4$, $Q = Q''$, $G' = G''/\text{center}$.

Since $Z_{G'}(N_1, N_2)$ is now of type A_1, there is a decomposition of the underlying space V' of G' into a direct sum $V' = V_1 \oplus V_2$, $\dim V_1 = \dim V_2 = 2$ where N_1, N_2 act as scalars on V_2 and N_3, N_4 act as scalars on V_1. Therefore we can represent elements of N_1 and N_2 as matrices of the form

$$\left(\begin{array}{cc|cc} a & b & \multicolumn{2}{c}{0} \\ c & d & & \\ \hline & & 1 & 0 \\ \multicolumn{2}{c|}{0} & 0 & 1 \end{array}\right)$$

(we can make the elements of N_1 and N_2 act as the identity on V_2 since they are defined only up to multiplication by a constant). Since $[N_1, N_2] = 1$ and because of 3.2.2(i) there are two possibilities: either $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$ for all the elements of N_1 and N_2, or $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ for all the elements of N_1 and N_2. In the second case the elements of N_1 and N_2 are semisimple. We now show the first case is impossible. Assume first that $a \neq 1$ and $b \neq 0$ for some element in N_1. Then

$$Z_{G'}(N_1) = \left\{ \left(\begin{array}{cc|cc} a & b & \multicolumn{2}{c}{0} \\ 0 & a & & \\ \hline \multicolumn{2}{c|}{0} & * & * \\ & & * & * \end{array} \right) \right\} \quad \text{and} \quad Z_{G'}DZ_{G'}(N_1) = \left\{ \left(\begin{array}{cc|cc} * & * & \multicolumn{2}{c}{0} \\ * & * & & \\ \hline \multicolumn{2}{c|}{0} & c & 0 \\ & & 0 & c \end{array} \right) \right\},$$

that is $DZ_{G'}(DZ_{G'}(N_1)) = DZ_{G'}(DZ_{G'}(N_1, N_2))$, a contradiction with 3.2.3(ii), (iv). Thus either $a = 1$ or $b = 0$ for each element of N_1 and N_2. If $b = 0$ for a non-trivial element of N_1, say, then by 3.2.2(i), $DZ_Q(N_1)$ contains the two non-central infinite commuting normal subgroups $DZ_Q(N_1, N_2)$ and $DZ_Q(N_3, N_4)$, contradicting 3.2.2(iv). Thus $b \neq 0$, and hence $a = 1$, for all non-trivial elements of N_1 and N_2. Therefore, $Z_{G'}(N_1) = Z_{G'}(N_2)$ (compare, for example, 2.3.4(i)) and hence $N_1 = N_2$ by 3.2.2(ii), contradicting 3.2.3(ii).

4.1.4. Now consider the case when n is odd. By an argument as in 4.1.3 we may reduce the problem to the case $n = 3$. If $m \leqslant 2$ then the connected component of every centralizer in G' is commutative and therefore cannot contain a quasi-fop-simple group. Hence $m = 3$. But then the only elements in G' for which the connected component of their centralizer is not solvable are elements from the groups $T \in X'$, and these elements are semisimple.

4.1.5. It remains to show that the groups $Z_{G'}DZ_{G'}(N)$ belong to X' for every $N \in X_Q$. Since we have already established that $m = n$, it follows from 3.2.3(ii) that $Z_{G'}(N_1, \ldots, N_i)$ is of type A_j, $j \leqslant n - i - 1$. Since they are distinct for $1 \leqslant i \leqslant n - 2$, we must have $j = n - i - 1$. Thus $Z_{G'}(N_1)$ is of type A_{n-2} and hence $Z_{G'}DZ_{G'}(N_1) \in X'$. To conclude the proof of 4.1.1, observe by 3.2.4(iii) that we may include $N = N_1$ in some chain.

4.1.6. THEOREM. *Let k' be a field, $G' = PGL_{k'}(m)$ and let (Q, X_Q, Y_Q) be a full triple of rank $n - 1$. If $m \leqslant n$ and $Q \subseteq G'(k')$ then*

(i) $m = n$,

(ii) $X_Q \subseteq X' = X(G')$ *and* $Y_Q \subseteq Y' = Y(G')$,

(iii) *the lines of $X_Q \cup Y_Q$ are contained in the lines of $X' \cup Y' \cong \mathbb{P}^{n-1}_{K'}$ where $K' = k' \oplus k'$,*

(iv) *the autoduality of $X' \cup Y'$ restricts to the autoduality of $X_Q \cup Y_Q$.*

We already know that $m = n$ and $X_Q \subseteq X'$. The rest of the proof will be broken into a number of steps.

4.1.7. We prove now that $Y_Q \subseteq Y'$. Let $U \in Y_Q$ and let $S, N_1, \ldots, N_n$ have the same meaning as in 3.2.5. Then it follows from the proof of 4.1.5 that $S' = DZ_{G'}(N_1, \ldots, N_{n-3})$ is of type A_2 (actually, it is G' if $m = 3$, and is isomorphic to $SL(3)$ if $m > 3$) and the group $Z_{G'}(N_1, \ldots, N_{n-2})$ is of type A_1.

Using 3.2.5(ii) we can conclude that any element $u \in U$ has one of the two forms

$$\left(\begin{array}{c|cc} aI_{n-2} & \multicolumn{2}{c}{0} \\ \hline 0 & \begin{matrix} b & c \\ 0 & b \end{matrix} \end{array}\right) \quad \text{or} \quad \left(\begin{array}{c|c} aI_{n-2} & 0 \\ \hline 0 & \begin{matrix} b & 0 \\ 0 & c \end{matrix} \end{array}\right)$$

with $a, b, c \in k'$.

If $u \in U$, $u \neq 1$, has the second form with a, b, c all distinct, then $Z_{G'}DZ_{G'}(u) \supseteq DZ_{G'}(N_1, ..., N_{n-2})$. It follows that $Z_Q DZ_Q(u) = U$ is not commutative, contradicting 3.2.5(i). If (exactly) two of the numbers a, b, c are equal, then $Z_{S'}(u)$ is isomorphic to $GL(2)$; but this group does not contain solvable subgroups of derived length 3 in contradiction with 3.2.5(iv). Thus if $u \in U$, $u \neq 1$, then u must have the first form. If now $c = 0$ we are again in the second form. Thus $c \neq 0$. If $a \neq b$ then $Z_{G'}DZ_{G'}(u) = DZ_{G'}(N_1, ..., N_{n-2})$ and hence $U = Z_Q DZ_Q(u)$ is not commutative. Therefore $a = b$ and now $Z_{G'}DZ_{G'}(u) \in Y' = Y(G')$, as required.

4.1.8. Now we show that the lines of $X_Q \cup Y_Q$ are contained in lines of $X' \cup Y'$. This is clear for regular lines. Consider a line $P \in \mathscr{P}_d(Q)$. Let $U \in Y_Q$ and $N_1, ..., N_n$ be a chain as used in 3.2.6 to define P. Then $Z_{G'}Z_{G'}DZ_{G'}(N_1, ..., N_{n-3}) = G'_{M'}$ where M' is a regular plane of $\mathbb{P}^{n-1}_{K'}(K')$. Since $Z_{G'}DZ_{G'}(U) \in Y'$ it follows that the points of M' corresponding to $H' \in X' \cup Y'$, with $H' \subset Z_{G'_{M'}}(U)$, belong to a degenerate line of M' (cf. 2.6.2). Finally, consider a line $P \in \mathscr{P}_i(Q)$. As above, we can establish (with notation $U_1, U_2, N_1, ..., N_n$ as in 3.2.6) that $Z_{G'}Z_{G'}DZ_{G'}(N_1, ..., N_{n-4}) = G'_{M'}$ where M' is a regular 3-dimensional subspace of $\mathbb{P}^{n-1}_{K'}(K')$. Also, $Z_{G'}Z_{G'}DZ_{G'}(N_1, ..., N_{n-2}) = G'_{R'}$ where R' is a regular line of $\mathbb{P}^{n-1}_{K'}(K')$, and the isotropic points L'_i, $i = 1, 2$, corresponding to $Z_{G'}DZ_{G'}(U_i)$ are in M' and $L'_2 \subset R'$. Since $[U_1, Z_{G'}Z_{G'}DZ_{G'}(N_1, ..., N_{n-2})] = 1$ it follows from 2.6.1 that L_1 and R' are orthogonal. Hence $(L'_1 \cup L'_2)^\perp = L'_1 + L'_2$ (where the orthogonal complement is taken in M') is a totally isotropic line of M' and our assertion for all lines follows.

4.1.9. The fact that the autoduality is the same follows from 2.5 together with 2.2.3(i) and 2.3.4(i).

4.1.10. THEOREM. *With the notation and hypotheses of* 4.1.6 *an imbedding* $Q \hookrightarrow PGL(m, k')$ *induces an imbedding of projective spaces which preserves lines and the autoduality. Moreover, this imbedding is an injective projectivity.*

Proof. Only the last assertion remains to be proved. This follows from the definition of a projectivity in 1.2.2 together with 1.1.7 and 2.6.1. The extra fact being used here is that if two points determine a line in the projective geometry of Q, then the injected images of these points in $\mathbb{P}^{n-1}_{K'}(K')$ determine a line, which is the image of the line from Q. While this is obvious over fields, it is not clear over $K' = k' \oplus k'$ and hence the need for 1.1.7 which also utilizes the available autoduality.

4.2. *Monomorphisms of Full Subgroups*

Let k and k' be infinite fields and K and K' be two-dimensional separable algebras over k and k', respectively, with $\mathrm{Gal}(K/k) = \{1, \sigma\}$ and $\mathrm{Gal}(K'/k') = \{1, \sigma'\}$. Let V and V' be non-degenerate Hermitian spaces over K and K' with forms F and F'. Set $\tilde{G} = GU(V, F)$ and $\tilde{G}' = GU(V', F')$; denote by ω and ω' the canonical projections $\omega: \tilde{G} \to G = PGU(V, F)$ and $\omega': \tilde{G}' \to G' = PGU(V', F')$. Let $C = C(\tilde{G})$ and S be a subgroup of $C(k)$. The subgroup Q of $\tilde{G}(k)/S$ is called full if $\omega(SQ)$ is full in $G(k)$, where in writing SQ we are abusing notation and this is to be understood as the preimage of Q in $\tilde{G}(k)$ (below, for $q \in Q$ we will also write Sq and again this is the preimage of q in $\tilde{G}(k)$).

4.2.1. THEOREM. *Let $\alpha: Q \to G'(k')$ be a homomorphism whose kernel belongs to the center of the full group Q. Then there exists a unique field monomorphism $\varphi: k \to k'$ and a unique central k'-homomorphism of algebraic groups $\beta: {}^{\varphi}\tilde{G} \to G'$ such that $\alpha(q) = \beta(\varphi^0(Sq))$ for $q \in Q$.*

The proof is included in that of the following related theorem.

4.2.2. THEOREM. *Under the assumptions of* 4.2.1 *there exists a unique monomorphism $\varphi: K \to K'$ such that $\varphi\sigma = \sigma'\varphi$ and a unique (up to a constant) φ-semilinear monomorphism $A: V \to V'$ such that $\mathbb{P}A \circ \omega(Sq) = \alpha(q) \circ \mathbb{P}A$ for $q \in Q$, where $\mathbb{P}A$ is the projectivity induced by A.*

Proof. We have from 4.1.10, after factoring out the center of $\tilde{G}$, an injective projectivity $\bar{\alpha}: \mathbb{P}_K^{n-1}(K) \to \mathbb{P}_{K'}^{n-1}(K')$ where $n = rk_K V$. Moreover, $\bar{\alpha}$ maps orthogonal regular points in $\mathbb{P}_K^{n-1}(K)$ into orthogonal regular points in $\mathbb{P}_{K'}^{n-1}(K')$ and consequently preserves the rank of regular subspaces (cf. 1.2.4). By the Fundamental Theorem of Projective Geometry there is a unique monomorphism $\varphi: K \to K'$ and a φ-semilinear injective mapping $A: V \to V'$ such that $\bar{\alpha} = \mathbb{P}A$, the projectivity induced by A. Since $\bar{\alpha}$ preserves the autoduality it follows that $F'(Ax, Ay) = \tau(A)F(x, y)^{\varphi}$ where $\tau = \tau(A) \in K^*$ and $x, y \in V$. Hence for $a \in K$ we have $a^{\sigma'\varphi}F'(Ax, Ay) = F'(Ax, A(ay)) = \tau F(x, ay)^{\varphi} = \tau a^{\varphi\sigma}F(x, y)^{\varphi}$. Thus $\varphi\sigma = \sigma'\varphi$ and therefore $\varphi(k) \subseteq k'$. Since $F'(Ax, Ay) = \tau F(x, y)^{\varphi}$ it follows that A induces a k'-isomorphism $\beta: {}^{\varphi}\tilde{G} \to G'$. It remains to prove that $\alpha(q) = \beta(\varphi^0(Sq))$, or equivalently, $\mathbb{P}A \circ \omega(Sq) = \alpha(q) \circ \mathbb{P}A$ for $q \in Q$. From the construction of $\bar{\alpha}$ we have for $N \in X_Q \cup Y_Q$ that $Z_{G'}(\bar{\alpha}(\omega(SN))) = Z_{G'}(\alpha(N))$. Take $q \in Q$ and $h \in N$ with $h \neq 1$. Then $\alpha(q)Z_{G'}(\alpha(h))\,\alpha(q)^{-1} = Z_{G'}(\alpha(qhq^{-1})) = Z_{G'}((\beta \circ \varphi^0)(Sqhq^{-1})) = \beta \circ \varphi^0(Sq)Z_{G'}(\beta \circ \varphi^0(Sh))\beta \circ \varphi^0(Sq)^{-1}$. It follows that $\alpha(q)$ and $\beta \circ \varphi^0(Sq)$ coincide as transformations on $\mathbb{P}_{K'}^{n-1}(K')$, as required.

4.2.3. THEOREM. *Let Q be a full subgroup of $\tilde{G}(k)$ (i.e., $S = \{1\}$) and let $\alpha: Q \to \tilde{G}'(k')$ be a group monomorphism. Then there exist a unique field monomorphism $\varphi: k \to k'$, a unique k'-isomorphism of algebraic groups $\beta: {}^{\varphi}\tilde{G} \to \tilde{G}'$ and a unique group homomorphism $\gamma: Q \to C(G'(k')) \cong K'^*$ such that $\alpha(q) = \beta(\varphi^0(q)) \circ \gamma(q)$ for $q \in Q$.*

As before we restate this theorem in the classical form.

4.2.4. THEOREM. *Under the assumptions of* 4.2.3. *there exist a unique monomorphism $\varphi: K \to K'$ such that $\varphi\sigma = \sigma'\varphi$, a unique (up to a constant) φ-semilinear monomorphism $A: V \to V'$ and a unique group homomorphism $\gamma: Q \to K'^*$ such that $A \circ q = (\gamma(q)\, \mathrm{Id}_{V'}) \circ \alpha(q) \circ A$ for all $q \in Q$.*

Proof. By 4.2.2 applied to $\omega' \circ \alpha$ there exist φ and A satisfying the required properties. Since $\mathbb{P}A \circ \omega(q) = \omega'(\alpha(q)) \circ \mathbb{P}A$ it follows that $A \circ q$ differs from $\alpha(q) \circ A$ only by a scalar factor from K'^*. Thus $A \circ q = (\gamma(q)\, \mathrm{Id}) \circ \alpha(q) \circ A$. It is immediate to check that $\gamma: Q \to K'^*$ is a group homomorphism.

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