15. Maximal subgroups of $\text{GL}_n(\mathbb{C})$.

Let $\psi_m: \text{Alt}_m \rightarrow \text{GL}_{m-1}(\mathbb{C})$ be the non-trivial component of the transitive representation of $\text{Alt}_m$ on $m$ letters and $\gamma_m: \text{Alt}_m \rightarrow \text{GL}_{m(m-2)/2}(\mathbb{C})$, the representation of $\text{Alt}_m$ such that the permutation representation (of degree $m(m-1)/2$) of $\text{Alt}_m$ on unordered pairs of distinct letters is equivalent to $\text{id} \oplus \psi_m \oplus \gamma_m$. 
We shall call a finite subgroup $G$ of $\text{GL}_n(\mathbb{C})$ nearly maximal if for any finite subgroup $H$ of $\text{GL}_n(\mathbb{C})$, the inclusion $H \leq G$ implies that $H \leq \text{NGL}_n(\mathbb{C})(G)$. In the cases we consider below, near-maximality of $G$ means that $\text{NGL}_n(\mathbb{C})(G)$ is, modulo its center, a maximal finite subgroup of $\text{GL}_n(\mathbb{C})$. But this interpretation does not hold in other examples.
(15.1) Proposition. There exists \( n_1 = n_1(r) \in \mathbb{N} \) such that if \( G := \otimes_{i=1}^r \varphi_{m_i}(\text{Alt}_{m_i}) \in \text{GL}_{\prod_{i=1}^r (m_i-1)}(\mathbb{C}) \) and \( m_i \geq n_1(r), \quad i = 1, \ldots, r \), then

\( G \) is nearly maximal in \( \text{GL}_{\prod_{i=1}^r (m_i-1)}(\mathbb{C}) \).

(15.2) Proposition. There exists \( n_2 \in \mathbb{N} \) such that if \( G := \psi_m(\text{Alt}_m) \in \text{GL}_{m(m-3)/2}(\mathbb{C}) \) and \( m \geq n_2 \), then \( G \) is nearly maximal in \( \text{GL}_{m(m-3)/2}(\mathbb{C}) \).
(15.3) Proposition. If \( G = \bigoplus_{i=1}^{r} \varphi_{m_i}(\text{Alt}_{m_i}) \subseteq \text{GL}_{\sum m_i (m_i - 1)}(\mathbb{C}) \) and \( m_i \geq 10 \), \( i = 1, \ldots, r \), then \( G \) is nearly maximal in \( \text{GL}_{\sum m_i (m_i - 1)}(\mathbb{C}) \).

(15.4) Proposition. There exists \( n_4 \in \mathbb{N} \) such that if \( G = \varphi_n(\text{Alt}_n) \oplus \varphi_n(\text{Alt}_n) \subseteq \text{GL}_{(m-1) + \frac{n(n-3)}{2}}(\mathbb{C}) \) and \( m \geq n_4 \), \( n \geq n_4 \), then \( G \) is nearly maximal in \( \text{GL}_{(m-1) + \frac{n(n-3)}{2}}(\mathbb{C}) \).
(15.5) Remark. (15.3) and (15.4) give examples of nearly maximal reducible finite linear groups.
(15.6) Proof of (9.51). First, if \( r = 1 \), then \( \tilde{G} = W_{m_1}(\text{Sym}_{m_1}) \) is a group generated by reflections. This \( G \) is known not to be maximal for \( m_1 = 9 \) (for \( \text{Sym}_9 \) is contained in the Weyl group of type \( E_8 \)). When \( m_1 > 9 \) consider \( H \supseteq G, H \subseteq \text{GL}_{m_1}(\mathbb{C}) \), \( H \) finite and then replace \( H \) by \( \tilde{H} = \langle H, \tilde{G} \rangle \). \( H \) contains a normal subgroup generated by reflections and from the classification of finite groups generated by reflections we see that \( G \) is nearly maximal if \( m_1 \geq 10 \) (so that \( \rho_1(1) = 10 \)).
Let us now choose \( n_1(n) \) so that
\[
n_1(n) \geq 49 \quad \text{and} \quad \left( \frac{n_1(n)}{2} \right)^5 > (2(n_1(n)-1)^5 + 1)
\]
Clearly, such \( n_1(n) \) exists and it is easy to see that \( n_1(n) > n_1(a) \) if \( a < n \).

Let now \( m := \pi^{-1} \circ (m; -1) \) and let \( H \leq G_{\ell m} \) be a finite group such that \( H \supseteq G \). \( H \) is primitive since so is \( G \). Let \( \overline{S} \) be the socle of \( H/\text{center} \) and \( S \) its preimage in \( H \). Then, as in Section 11, \( S \) is a central product of centrally simple groups \( G_1, \ldots, G_t \), extraspecial groups \( E, \ldots, E_s \), and of the center \( C \) of \( H \). We have, of course,
need to be proven

\[ G \subseteq S \text{ whence at once } s = 0. \text{ We show now that each } \psi_n, (\text{Alt}_m, \sigma) \text{ is contained in some } G_j. \text{ Suppose that is not so. Assume, for definiteness, that } G_1 \text{ is such that the projections of } \psi_n, (\text{Alt}_m, \sigma) \text{ on } G_j \text{ and on } G_2, \ldots, G_t \text{ are non-trivial. The representation of the central product of } G_1 \text{ and } G_2, \ldots, G_t \text{ on } k^m \text{ is equivalent to the tensor product of representations } \otimes_k \text{ of the two factors. Restricted to } \psi_n, (\text{Alt}_m, \sigma) \text{ it implies that } \psi_n \text{ is a tensor product of two representations of Alt}_m. \text{ But, since } \psi_n \text{ is a non-trivial representation of smallest dimension, it can not be a tensor product. Thus each } \psi_n, (\text{Alt}_m, \sigma) \text{ is contained in some } G_j. \]
Thus \( r \leq U_{i=1}^t I_i \) with \( I_i \cap I_j = \emptyset \) if \( i \neq j \) and \( G_j = \bigotimes_{c \in I_j} \Phi_{m_i}(A_{tm_i}) \otimes d_i \).

Let us now argue by induction on \( r \). The case \( r = 1 \) was dealt with before. Since \( n_1(r) \geq n_1(|I_j|) \) for \( j = 1, \ldots, t \), it follows \( |I_j| = 1 \) for \( j = 1, \ldots, t \) (and then \( t = r \), whence \( H = N_{G\leq_m}(G) \)) if \( t = 1 \). Our choice of \( n_1(r) \), together with \((61),(71)\) implies then that

\( G_1 \cong \text{Alt}_d \) for some \( d \leq m + 1 \).

Let \( \Psi_i \) denote the embedding of \( \text{Alt}_{m_i} \) in \( \text{Alt}_d \). Let \( \Omega_1, \ldots, \Omega_{2b} \) be different
orbits of \( \varphi_i(\text{Alt}_{m_i}) \) on \( 1, \ldots, d_i \). Let 
\( \{1, \ldots, 6\} = \bigcup_{i=1}^c J_i \) so that the orbits 
\( \Omega_{\alpha} \) and \( \Omega_{\beta} \) are equivalent if and
only if \( \alpha, \beta \in J_i \) for some \( i \). Then 
\[ Z_{\text{Alt}_d}(\varphi_i(\text{Alt}_{m_i})) \cong \prod_{i=1}^c \text{Sym } J_i \]
where each \( \text{Sym } J_i \) permutes the orbits \( \Omega_{\alpha} \), \( \alpha \in J_i \).
Since \( \varphi_i(\text{Alt}_{m_i}) \leq Z_{\text{Alt}_d}(\varphi_i(\text{Alt}_{m_i})) \) for 
\( i=2, \ldots, r \) and by the inductive assumption 
we must have \( \varphi_i(\text{Alt}_{m_i}) = \text{Sym } J_{\alpha(i)} \) 
for \( i=2, \ldots, r \) and an appropriate \( \alpha(i)=1, \ldots, c \).
Note that $\text{Alt } J_1$ acts trivially on $U \cap J_1$. This implies that each $\text{Alt } J_i$ has at most two types of orbits on $\{1, \ldots, d_i\}$; and if exactly two then one type is trivial. By symmetry this therefore holds for all $\varphi_i(\text{Alt } m_i)$, and in particular, for $i = 1$. Thus $c \leq 2, r \leq 3$.

Suppose there is a trivial orbit, say $\Omega_1$, of $\varphi_1(\text{Alt } m_1)$. Let $\Omega_1 \in J_1$. Then $\mathbb{Z}_{\text{Alt } J_1}(\text{Alt } J_1) = \text{Alt}(\{1, \ldots, d_1 - J_1\})$ whence again by inductive assumption we must have $\mathbb{Z}_{\text{Alt } J_1}(\text{Alt } J_1)$ is one of the $\varphi_i(\text{Alt } m_i)$ (when there is a trivial orbit).

Thus we can assume in this case that $c = r = 2$, and $\varphi_i(\text{Alt } m_i)$ acts through the natural representation on its non-trivial orbit. We have thus $d = m_1 + m_2$. Then
\[ m = (m_1 - 1)(m_2 - 1) \leq \left( \frac{d-2}{2} \right)^2. \] Thus we are dealing with a representation of \( \text{Alt}_d \) of dimension \( \leq \left( \frac{d-2}{2} \right)^2 \). By R. Rasala [p.132, Result 2] this implies (since \( d \geq 2n_1(2) \geq 20 \)) that \( m \leq d-1 \), i.e.

\[(m_1 - 1)(m_2 - 1) \leq m_1 + m_2 - 1 \quad \text{or} \quad (m_1 - 2)(m_2 - 2) \leq 2.\]

This latter inequality is false for \( m_1, m_2 \geq 10 \). Thus our current assumption that there are trivial orbits is false as well.
Thus there are no trivial orbits and therefore $J_1 = \{1, \ldots, d\}$. We assume $r > 1$, so that $r = 2$. Since $\varphi_2 (\text{Alt}_m)$ acts as $\text{Alt} J_1$ on the non-trivial orbits, we see that non-trivial orbits of $\varphi_i (\text{Alt}_m)$ are of length $m_i$ for $i = 1, 2$. Thus $d = m_1 m_2$. Since $m = (m_1 - 1)(m_2 - 1) < d - 1$ it follows that $\text{Alt}_d$ can have no representation of dimension $m$, whence a contradiction in this case. This concludes the proof of (15.1).
(15.7) Proof of (15.2). Again, take \( n_2 > 49 \) and 
\[
\left( \frac{n_2!}{2} \right)^2 > \left( n_2(n_2-3)+1 \right)
\]
such that \((n_2)!/2 > (n_2(n_2-3)+1)\).

Clearly, such \( n_2 \) exists. Set \( n := n(n-3)/2 \).

Let \( H \in \text{GL}_n(C) \), \( H \) finite, \( H \not\subset G \).

Since \( G \) is primitive so is \( H \). Let \( S \) be the preimage in \( H \) of the socle of \( H/\text{center} \). Then \( S \) is a central product of centrally simple groups \( G_1, \ldots, G_t \), extraspecial groups \( E_1, \ldots, E_s \), and of the center \( C \) of \( H \). We have again that \( s = 0 \). Clearly, \( G = G_1 \ldots G_t \).

Let \( G_i(x) \) be the projection of \( G \) on \( G_i \), \( i = 1, \ldots, t \). The representation of \( G_1 \ldots G_t \) on \( k^n \) is a tensor product \( \bigotimes_{i=1}^t V_i \) of representations of the \( G_i \). Therefore, our representation
of $G = \text{Alt}_m$ is the tensor product of
the $\pi_i |G_{(i)}|$. But since by R. Rasala [1] non-trivial
the smallest representation of $G$ has
dimension $m-1$ and since $(m-1)^2 > m(m-1)/3$
it follows that $\gamma_m$ is not a tensor
product whence $t = 1$.

Then by (6.1), (7.1) and because of our
choice of $\nu_2$ we see that $G_1 \cong \text{Alt}_d$
for some $d$. We have since $\text{Alt}_d \supset \text{Alt}_m$
that $d \geq m$. If $d > m$ then $\frac{d(d-3)}{2} > m$
and hence by R. Rasala [1, Result 2] we have

\[ n = d-1 \quad \text{i.e.} \quad d = m(m-3)/2 + 1. \]
Consider the action of $\text{Alt}_m$ on $\mathcal{S}_2$.

If $\text{Alt}_m$ has a fixed point on $\mathcal{S}_2$ then $\text{Alt}_m \leq \text{Alt}_{d-1}$ and the restriction of our representation of $\text{Alt}_d$ (of dimension $d-1$) to $\text{Alt}_{d-1}$ would be irreducible which is not the case. If $\text{Alt}_m$ has an orbit $\mathcal{S}_1 \neq \emptyset$, then $\text{Alt}_m = \text{Alt}_{\mathcal{S}_1} \times \text{Alt}(\mathcal{S}_2 - \mathcal{S}_1)$ and the restriction of our representation of $\text{Alt}_d$ (of dimension $d-1$) to $\text{Alt}_{\mathcal{S}_1} \times \text{Alt}(\mathcal{S}_2 - \mathcal{S}_1)$ would be irreducible. But this is not so if $\mathcal{S}_1 \neq \mathcal{S}_2$, $\mathcal{S}_2 \neq \emptyset$. Thus $\text{Alt}_m$ is transitive on $\mathcal{S}_2$.

Since any primitive permutation representation of $\text{Alt}_m$ has degree $m \geq m(m-1)/2$ (see...
it follows \( \sqrt{\frac{m(m-3)}{2}} + 1 < \frac{m(m-1)}{2} \) that

the orbit of Alt \( m \) on \( S^2 \) has

length divisible by \( m \). Thus \( m(m-3)/2 + 1 = rm \)

for some \( r \in \mathbb{N} \). Clearly \( r > (m-3)/2 \). Thus

\[ 2 = m(2r - (m-3)) \quad \text{with} \quad 2r - (m-3) > 1. \]

This is clearly impossible for \( m \geq 49 \). This

is a contradiction with the assumption \( d > m \).

Thus \( d = m \) and \( (15, 2) \) is proved.
(15.8) Proof of (15.3). For $r = 1$, (15.3) reduces to (15.1). So assume $r \geq 2$.

Let $m := \sum_{i=1}^{r} (m_{i}-1)$ and let $H \leq GL_{m}(C)$ be a finite group such that $H \leq G$. Replace $H$ by $\langle H, \Theta_{i}, \gamma_{m_{i}}(\text{Symm}_{m_{i}}) \rangle$ and let $M$ be the normal subgroup of $H$ generated by $\Theta_{i}, \gamma_{m_{i}}(\text{Symm}_{m_{i}})$. Set $V := k^{m}$ and let $V = \oplus_{i=1}^{r} V_{i}$ be the decomposition of $V$ into $M$-simple modules. Then $M_{i} = M \cap V_{i}$ is generated by reflections and is irreducible. Then by classification
of finite irreducible groups generated by reflections (see, e.g., and since \( \dim V_i = \min(n_i - 1) \geq 9 \) we have that \( M_i \) is a Weyl group of one of the types \( A_5, B_5, \) or \( D_5 \). We can assume that \( M_i = M, \ V_i = V \). Thus we obtain in case \( A_5 \) that \( \operatorname{Alt}_{n+1} \) contains a direct product of \( \operatorname{Alt}_n \) with \( \Sigma m_i = m+r \), and in the case of \( B_5 \) and \( D_5 \), by taking quotient \( M \) by its radical, that \( \operatorname{Alt}_n \) contains a direct product of \( \operatorname{Alt}_n \) with \( \Sigma m_i = m+r \).

This situation is easily seen to be impossible unless \( r = 1 \), \( M \) is of type \( A_{n+1} \). Returning to our original \( M \) this means that each \( V_i \) is irreducible for exactly one \( \varphi_i \) of \( \operatorname{Alt}_{n+1} \) whence our claim.
(15.9) Proof of (15.9). Let us take \( n_1 \geq \max(n_1,1), n_2 \) and such that 
\[
(2n_1 + n_1(n_1-3)+1) \geq 2 \log_3(2n_1 + n_1(n_1-3)+1) + 1
\]

Let \( d = n-1 + n(n-3)/2 \) and let \( H \leq \text{GL}_d(k) \), \( H \) finite, \( H \geq G \). If \( H \) is reducible then the irreducible components clearly will have dimensions \( n-1 \) and \( n(n-3)/2 \) and then \( H \leq N_{\text{GL}_d(k)}(G) \) by (15.1) and (15.2). If \( H \) is irreducible it is easily seen to be primitive. Let \( S \) be the preimage in \( H \) of the image of \( H/\text{centre} \). As before one shows that \( S \) must be centrally simple and then, by (6.1), (7.1), \( S \cong \text{Alt}_r \), for some \( r \). One has at once, looking at lengths of
orbits of $\text{Alt}_n$ and $\text{Alt}_m$ on $\mathcal{S}_2 = \{1, \ldots, r\}$ that $r \geq m + n$. Using again R. Rasala's Result 2, and noting that

$$m-1 + n(n-3)/2 \leq (m+n)(m+n-3)/2 \leq r(r-3)/2$$

we see that $r = d+1 = m + n(n-3)/2$.

Let $\mathcal{S}_2, \ldots, \mathcal{S}_k$ be different orbits of $\text{Alt}_m$ on $\mathcal{S}_2$. Since the representation of $\text{Alt}_r$ on $k$ is the smallest one and since its restriction to $\text{Alt}_m$ is again the smallest one taken once plus a number of trivial ones, we have that one orbit, say $\mathcal{S}_1$, is of length 1 and the rest are of length 1. Then $Z_{\text{Alt}_r} (\text{Alt}_m) = \text{Alt}_{r-m}$ whence $\text{Alt}_n \leq \text{Alt}_{r-m}$ which contradicts near maximality of $\psi_r(\text{Alt}_n)$ (i.e. contradicts (15.2)) unless $r-m = n$. But this latter
variant is impossible as $n < n(n-3)/2$. 